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WEIGHTED RELLICH TYPE INEQUALITIES RELATED TO BAOUENDI-GRUSHIN OPERATORS

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Abstract. We find a simple sufficient criterion on a pair of nonnegative weight functions $a(x,y)$ and $b(x,y)$ in $\mathbb{R}^{m+k}$ so that the general weighted L^p Rellich type inequality

$$\int_{\mathbb{R}^{m+k}} a(x,y)\,|\Delta_\gamma u(x,y)|^p\,dxdy \geq \int_{\mathbb{R}^{m+k}} b(x,y)\,|u(x,y)|^p\,dxdy$$

holds for all $u \in C_0^\infty(\mathbb{R}^{m+k})$. Here $\Delta_\gamma = \Delta_x + |x|^{2\gamma}\Delta_y$ is the Baouendi-Grushin operator, $\gamma > 0$, $m, k \geq 1$ and $p > 1$. It is important to point out here that our approach is constructive in the sense that it allows us to retrieve already established weighted sharp Rellich type inequalities as well as to get other new results with an explicit constant on $\mathbb{R}^{m+k}$. We also obtain a sharp L^p Rellich type inequality that connects first to second order derivatives and several new two-weight Rellich type inequalities with remainder terms on smooth bounded domains Ω in $\mathbb{R}^{m+k}$ via a nonlinear differential inequality.

1. Introduction

The classical Rellich inequality [18] states that for $n \geq 5$

$$\int_{\mathbb{R}^n} |\Delta u|^2 dz \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{|z|^4} dz, \tag{1.1}$$

where $u \in C_0^\infty(\mathbb{R}^n)$ and the constant $\frac{n^2(n-4)^2}{16}$ is sharp. Here, $\Delta, |\cdot|$ and dz denote the classical Laplace operator, the Euclidean norm and the Lebesgue measure on $\mathbb{R}^n$, respectively.

After Rellich's proof, many studies have been devoted to extensions and improvements of (1.1) on the Euclidean space $\mathbb{R}^n$. For instance, Davies and Hinz [6] extended the inequality (1.1) to the L^p case and showed that for any $1 < p < \frac{n}{2}$ and $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ there holds

$$\int_{\mathbb{R}^n} |\Delta u|^p\,dz \geq \frac{(n-2p)^p\,(np-n)^p}{p^{2p}} \int_{\mathbb{R}^n} \frac{|u|^p}{|z|^{2p}} dz, \tag{1.2}$$

where the constant $\frac{(n-2p)^p(np-n)^p}{p^{2p}}$ is sharp.

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Tertikas and Zographopoulos [19] obtained the following Rellich type inequality that connects first to second order derivatives:

$$\int_{\mathbb{R}^n} |\Delta u|^2 dz \geq \frac{n^2}{4} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|z|^2} dz, \tag{1.3}$$

where $u \in C_0^\infty(\mathbb{R}^n)$, $n \geq 5$ and the constant $\frac{n^2}{4}$ is sharp.

On the other hand, the lack of extremals in (1.1) , (1.2) and (1.3) has inspired many mathematicians to consider possible extra nonnegative terms on the right hand sides. Interesting results in this direction were established by many authors, and we refer the reader to the articles [2], [3], [19], [8], [15], [5], [16] and the references therein.

However, Euclidean results mentioned above continue to be a source of inspiration for the problem of finding analogues and new inequalities on the sub-Riemannian space $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ defined by the Baouendi-Grushin vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, m, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, k,$$

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $\gamma > 0$. We should say that the literature for Rellich inequalities related to Baouendi-Grushin operator is more restricted compared to the Euclidean case. In this context, Kombe [12] established the following weighted L^2 Rellich inequality for any $u \in C_0^\infty(\mathbb{R}^{m+k} \setminus \{(0,0)\})$:

$$\int_{\mathbb{R}^{m+k}} \rho^\alpha \frac{\rho^{2\gamma}}{|x|^{2\gamma}} |\Delta_\gamma u|^2 dxdy \geq \frac{(Q+\alpha-4)^2 (Q-\alpha)^2}{16} \int_{\mathbb{R}^{m+k}} \rho^{\alpha-4} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} u^2 dxdy, \tag{1.4}$$

provided that $\alpha \in \mathbb{R}$, $Q+\alpha-4 > 0$ and $Q > \alpha$. Here $Q := m+(1+\gamma)k$ is the homogeneous dimension of $\mathbb{R}^{m+k}$ with $m, k \geq 1$ and $\rho := \left(|x|^{2(1+\gamma)} + (1+\gamma)^2 |y|^2\right)^{\frac{1}{2(1+\gamma)}}$ is the *gauge* function associated with the fundamental solution for the sub-elliptic operator $\Delta_\gamma := \Delta_x + |x|^{2\gamma} \Delta_y$. Moreover, the positive constant appearing in (1.4) is sharp.

Recently, Shen and Jin [11] have generalized the inequality (1.4) to the nonlinear case $p \neq 2$ and proved that for any $u \in C_0^\infty\left(\mathbb{R}^{m+k} \setminus \{(0,0)\}\right)$, one has

$$\int_{\mathbb{R}^{m+k}} \rho^{\alpha+2p} \frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} |\Delta_\gamma u|^p dxdy \geq \widetilde{c}\,(Q, \alpha, p) \int_{\mathbb{R}^{m+k}} \rho^\alpha \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p dxdy \tag{1.5}$$

whenever $1 < p < \infty$ and $-Q < \alpha < (p-1)(Q-2) - 2$. Furthermore, the positive constant $\widetilde{c}\,(Q, \alpha, p) := (\frac{Q+\alpha}{p})^p (\frac{Q(p-1)-2p-\alpha}{p})^p$ in (1.5) is sharp.

Even though the literature has mostly focused on power radial weights $\rho^\alpha |x|^\beta$ in the inequalities above, it is natural to investigate these inequalities with non-radial weights. In this line of research, our first goal is to establish a constructive method that derives L^p Rellich type inequalities with general weights on the sub-Riemannian space $\mathbb{R}^{m+k}$ defined by the Baouendi-Grushin vector fields. More precisely, we show that if $a \in C^2\left(\mathbb{R}^{m+k}\right)$ and $b \in L^1_{loc}\left(\mathbb{R}^{m+k}\right)$ are nonnegative functions and $\vartheta \in C^\infty\left(\mathbb{R}^{m+k}\right)$ is a positive function such that

$$\Delta_\gamma \left(a\, |\Delta_\gamma \vartheta|^{p-2} \Delta_\gamma \vartheta\right) \geq b \vartheta^{p-1} \quad \text{and} \quad -\Delta_\gamma \vartheta > 0 \tag{1.6}$$

almost everywhere in $\mathbb{R}^{m+k}$, then the general weighted L^p Rellich inequality

$$\int_{\mathbb{R}^{m+k}} a\,|\Delta_\gamma u|^p\,dxdy \geq \int_{\mathbb{R}^{m+k}} b\,|u|^p\,dxdy$$

holds for all $u \in C_0^\infty\left(\mathbb{R}^{m+k}\right)$ and $p > 1$. It is worthwhile to notice here that our approach is efficient in the sense that by making proper choices for a and ϑ in the inequalities (1.6) it allows us to retrieve previously known weighted sharp Rellich type inequalities as well as to derive other new results with an explicit constant on some special domains in $\mathbb{R}^{m+k}$ (see Applications of Theorem 3.2).

The second goal of this paper is to obtain a sharp weighted L^p analogue of the inequality (1.3) on the sub-Riemannian space $\mathbb{R}^{m+k}$ defined by the Baouendi-Grushin vector fields. The third goal of this paper is to present some new results on two-weight L^p Rellich type inequalities with remainder terms on smooth bounded domains Ω in $\mathbb{R}^{m+k}$. The primary tools which we employ in constructing an improved two-weight Rellich type inequality are the improved two-weight Hardy type inequality appearing in [13] and a differential inequality involving a nonnegative general weight function a, the *gauge* function ρ and a positive smooth function δ (see Theorem 5.1). By specializing the functions a and δ in Theorem 5.1, we give several concrete examples including radial, logarithmic and exponential weights (see Applications of Theorem 5.1).

2. Preliminaries

In this section, we shall collect some notation, definitions and preliminary facts, which will be used throughout the article. The generic point is $z = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ with $m, k \geq 1$ and $m + k = n$. The sub-elliptic gradient is the n dimensional vector field given by

$$\nabla_\gamma = (\nabla_x, |x|^\gamma \nabla_y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_k),$$

where

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, m, \quad Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, k. \tag{2.1}$$

The Baouendi-Grushin operator on $\mathbb{R}^{m+k}$ is the operator

$$\Delta_\gamma = \nabla_\gamma \cdot \nabla_\gamma = \Delta_x + |x|^{2\gamma} \Delta_y,$$

where Δ_x and Δ_y are the classical Laplace operators in the variables $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^k$, respectively (see [1], [9], [10]). If γ is an even positive integer, then Δ_γ is a sum of squares of C^∞ vector fields satisfying the Hörmander finite rank condition: *rank Lie* $[X_1, \ldots, X_m, Y_1, \ldots, Y_k] \equiv n$. The anisotropic dilation attached to Δ_γ is given by

$$\delta_\lambda(z) = (\lambda x, \lambda^{\gamma+1} y), \quad \lambda > 0, \quad z = (x, y) \in \mathbb{R}^{m+k}. \tag{2.2}$$

It is easy to verify that

$$d\delta_\lambda(x, y) = \lambda^Q dxdy,$$

where

$$Q = m + (1 + \gamma)k$$

is the homogeneous dimension with respect to the dilation δ_λ and $dxdy$ denotes the Lebesgue measure on $\mathbb{R}^{m+k}$. Let us recall the definition of *gauge* ρ associated to the sub-elliptic operator Δ_γ. For $z=(x,y)\in\mathbb{R}^m\times\mathbb{R}^k$,

$$\rho=\rho(z):=\left(|x|^{2(1+\gamma)}+(1+\gamma)^2|y|^2\right)^{\frac{1}{2(1+\gamma)}}.$$

We remark that ρ is strictly positive, smooth away from the origin and symmetric. It is not hard to see that ρ is homogeneous of degree one with respect to the anisotropic dilations (2.2), that is, $\rho(\delta_\lambda(z))=\lambda\rho(z)$ for every $\lambda>0$. The sub-elliptic gradient of the *gauge* ρ satisfies

$$\nabla_\gamma\rho=\Big(\frac{|x|^{2\gamma}x}{\rho^{2\gamma+1}},\frac{(1+\gamma)|x|^\gamma y}{\rho^{2\gamma+1}}\Big)$$

and we get

$$|\nabla_\gamma\rho|=\frac{|x|^\gamma}{\rho^\gamma}. \tag{2.3}$$

Let $u\in C^2$ and define $u=u(\rho)$; then we have the following useful formula:

$$\Delta_{\gamma,p}u=\frac{|x|^{\gamma p}}{\rho^{\gamma p}}\left|u'(\rho)\right|^{p-2}\left[(p-1)\,u''(\rho)+(Q-1)\,\frac{u'(\rho)}{\rho}\right]$$

at every point $z\in\mathbb{R}^n-\{0\}$ with $u'(\rho(z))\neq 0$. This formula shows that for a suitable constant $C_{\gamma,Q}>0$ the function $\Gamma=C_{\gamma,Q}\rho^{2-Q}$ is a fundamental solution of $-\Delta_\gamma$ with pole at 0 (see [7]).

Using the above formulas one can easily obtain the following identities:

$$\nabla_\gamma\rho^\alpha\cdot\nabla_\gamma|x|^\beta=\alpha\beta\frac{|x|^{2\gamma}}{\rho^{2\gamma}}|x|^\beta\rho^{\alpha-2} \tag{2.4}$$

and

$$\Delta_\gamma\left(\rho^\alpha|x|^\beta\right)=\alpha\left(Q+2\beta+\alpha-2\right)\frac{|x|^{2\gamma}}{\rho^{2\gamma}}|x|^\beta\rho^{\alpha-2}+\beta\left(\beta+m-2\right)\rho^\alpha|x|^{\beta-2} \tag{2.5}$$

with $\alpha,\ \beta\in\mathbb{R}$ and $m\geq 1$. Another remarkable property is that the *gauge* ρ is infinite harmonic in $\mathbb{R}^n-\{0\}$ (see [4]), that is, *gauge* ρ is the solution of

$$\nabla_\gamma(|\nabla_\gamma\rho|^2)\cdot\nabla_\gamma\rho=0. \tag{2.6}$$

We let $B_R=\{z\in\mathbb{R}^n\mid\rho(z)<R\}$, $\partial B_R=\{z\in\mathbb{R}^n\mid\rho(z)=R\}$ and call these sets respectively ρ-ball and ρ-sphere centered at the origin with radius R.

3. General weighted Rellich type inequalities

Before proving our main result in this section, we now give the following Picone type identity for the p-biharmonic operators in $\mathbb{R}^n$ that will be used in the sequel.

Lemma 3.1. *Let $u\geq 0$ and $\vartheta>0$ be smooth functions such that $-\Delta_\gamma\vartheta>0$ in $\mathbb{R}^n$. Denote*

$$R(u,\vartheta)=|\Delta_\gamma u|^p-\Delta_\gamma\left(\frac{u^p}{\vartheta^{p-1}}\right)|\Delta_\gamma\vartheta|^{p-2}\Delta_\gamma\vartheta$$

and

$$L(u,\vartheta) = |\Delta_\gamma u|^p - p\frac{u^{p-1}}{\vartheta^{p-1}}\Delta_\gamma u\,|\Delta_\gamma\vartheta|^{p-2}\,\Delta_\gamma\vartheta + (p-1)\,\frac{u^p}{\vartheta^p}\,|\Delta_\gamma\vartheta|^p$$
$$- p(p-1)\,\frac{u^{p-2}}{\vartheta^{p-1}}\,|\Delta_\gamma\vartheta|^{p-2}\,\Delta_\gamma\vartheta\left(\nabla_\gamma u - \frac{u}{\vartheta}\nabla_\gamma\vartheta\right)^2,$$

where $p>1$. *Then it follows that*

$$L(u,\vartheta) = R(u,\vartheta) \geq 0.$$

Proof. The proof is quite standard and similar to the Heisenberg group case proved by Niu et al. (see Lemma 3.1 in [17]). □

We are now ready to state the main theorem of this section.

Theorem 3.2. *Let* $a\in C^2(\mathbb{R}^n)$ *and* $b\in L^1_{loc}(\mathbb{R}^n)$ *be nonnegative functions and* $p>1$. *Suppose that* $\vartheta\in C^\infty(\mathbb{R}^n)$ *is a positive function satisfying the differential inequalities*

$$\Delta_\gamma\left(a\,|\Delta_\gamma\vartheta|^{p-2}\,\Delta_\gamma\vartheta\right)\geq b\vartheta^{p-1}\quad \textit{and}\quad -\Delta_\gamma\vartheta>0$$

almost everywhere in $\mathbb{R}^n$. *Then for any* $u\in C_0^\infty(\mathbb{R}^n)$, *there holds*

$$\int_{\mathbb{R}^n} a\,|\Delta_\gamma u|^p\,dz \geq \int_{\mathbb{R}^n} b\,|u|^p\,dz.$$

Proof. We first consider $0\leq u\in C_0^\infty(\mathbb{R}^n)$. It follows from Lemma 3.1 and the integration by parts formula that

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^n} aL(u,\vartheta)\,dz = \int_{\mathbb{R}^n} aR(u,\vartheta)\,dz\\
&= \int_{\mathbb{R}^n} a\,|\Delta_\gamma u|^p\,dz - \int_{\mathbb{R}^n} a\Delta_\gamma\left(\frac{u^p}{\vartheta^{p-1}}\right)|\Delta_\gamma\vartheta|^{p-2}\,\Delta_\gamma\vartheta dz\\
&= \int_{\mathbb{R}^n} a\,|\Delta_\gamma u|^p\,dz - \int_{\mathbb{R}^n} \frac{u^p}{\vartheta^{p-1}}\Delta_\gamma\left(a\,|\Delta_\gamma\vartheta|^{p-2}\,\Delta_\gamma\vartheta\right)dz\\
&\leq \int_{\mathbb{R}^n} a\,|\Delta_\gamma u|^p\,dz - \int_{\mathbb{R}^n} bu^p dz,
\end{aligned}$$

where we have used the differential inequality

$$\Delta_\gamma\left(a\,|\Delta_\gamma\vartheta|^{p-2}\,\Delta_\gamma\vartheta\right)\geq b\vartheta^{p-1}.$$

Therefore, for any nonnegative function $u\in C_0^\infty(\mathbb{R}^n)$ the following inequality:

$$\int_{\mathbb{R}^n} a\,|\Delta_\gamma u|^p\,dz \geq \int_{\mathbb{R}^n} b\,|u|^p\,dz$$

is valid. For general u, by letting $u=u^+-u^-$, we complete the proof. □

Applications of Theorem 3.2. Let $\epsilon>0$ be given. To make the following arguments rigorous we should replace the function ρ with its regularization

$$\rho_\epsilon := \left(|x|_\epsilon^{2(1+\gamma)} + (1+\gamma)^2\sum_{i=1}^k y_i^2\right)^{\frac{1}{2(1+\gamma)}},$$

where $|x|_\epsilon := \left(\epsilon^2+\sum_{i=1}^m x_i^2\right)^{1/2}$ and let $\epsilon\longrightarrow 0$ in the end. However, for the sake of simplicity we will proceed formally.

As we mentioned earlier our method is flexible enough to establish new weighted Rellich type inequalities and recover previously known results. We now demonstrate these cases by the following concrete examples. As a first example, let us consider the pair

$$a=\frac{\rho^{\alpha+2\gamma}}{|x|^{2\gamma}} \quad \text{and} \quad \vartheta=\rho^{-\left(\frac{Q+\alpha-4}{2}\right)}$$

in Theorem 3.2; then we have the following result due to Kombe [12].

Corollary 3.1. *Let $\alpha\in\mathbb{R}$, $Q+\alpha-4>0$ and $Q>\alpha$. Then for every $u\in C_0^{\infty}(\mathbb{R}^n)$, there holds*

$$\int_{\mathbb{R}^n}\rho^{\alpha}\frac{\rho^{2\gamma}}{|x|^{2\gamma}}\left|\Delta_{\gamma}u\right|^2dz\geq\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}\int_{\mathbb{R}^n}\rho^{\alpha-4}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}u^2dz.$$

Moreover, the constant $\frac{(Q+\alpha-4)^2(Q-\alpha)^2}{16}$ is sharp.

We now set the pair as

$$a=\rho^{\alpha+2p}\frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} \quad \text{and} \quad \vartheta=\rho^{-\left(\frac{Q+\alpha}{p}\right)};$$

then we have the weighted L^p Rellich inequality proved in [11] by Shen and Jin.

Corollary 3.2. *Let $1<p<\infty$ and $-Q<\alpha<(p-1)(Q-2)-2$. Then for every $u\in C_0^{\infty}(\mathbb{R}^n\setminus\{0\})$, there holds*

$$\int_{\mathbb{R}^n}\rho^{\alpha+2p}\frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}}\left|\Delta_{\gamma}u\right|^p dz\geq\widetilde{c}(Q,\alpha,p)\int_{\mathbb{R}^n}\rho^{\alpha}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}|u|^p dz,$$

where $\widetilde{c}(Q,\alpha,p):=(\frac{Q+\alpha}{p})^p(\frac{Q(p-1)-2p-\alpha}{p})^p$.

One can however apply Theorem 3.2 to obtain other new inequalities on some special domains in $\mathbb{R}^n$. For instance, specializing the functions as

$$a=\rho^{\alpha}\frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} \quad \text{and} \quad \vartheta=\log\frac{R}{\rho}, \quad R>\sup_{z\in\Omega}\rho$$

on a smooth bounded domain Ω in $\mathbb{R}^n$, we get the following inequality.

Corollary 3.3. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Let $\alpha\geq 0$, $\frac{Q+\alpha}{2}>p>\frac{\alpha+2}{2}$ and $R>\sup_{z\in\Omega}\rho$. Then for every $u\in C_0^{\infty}(\Omega)$, we have*

$$\int_{\Omega}\rho^{\alpha}\frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}}\left|\Delta_{\gamma}u\right|^p dz\geq c_0(Q,\alpha,p)\int_{\Omega}\frac{\rho^{\alpha}}{\rho^{2p}\left(\log\frac{R}{\rho}\right)^{p-1}}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}|u|^p dz,$$

where $c_0(Q,\alpha,p):=(2p-\alpha-2)(Q+\alpha-2p)(Q-2)^{p-1}$.

Now let us take the pair as

$$a=\rho^{\alpha}\frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} \quad \text{and} \quad \vartheta=R-\rho$$

on the $\rho-$ball B_R in $\mathbb{R}^n$. Then we have the following result.

Corollary 3.4. *Let $Q+\alpha-1>p>\alpha+1$ and $\alpha\geq 0$. Then for every $u\in C_0^\infty(B_R)$, one has*

$$\int_{B_R}\rho^\alpha\frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}}\left|\Delta_\gamma u\right|^p dz\geq c_1(Q,\alpha,p)\int_{B_R}\frac{\rho^{\alpha-p-1}}{(R-\rho)^{p-1}}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}|u|^p dz,$$

where $c_1(Q,\alpha,p):=(p-\alpha-1)(Q+\alpha-p-1)(Q-1)^{p-1}$ and B_R is the ρ -ball.

On the other hand, considering the nonradial functions

$$a=x_1^{2p-2}\log y_1\quad\text{and}\quad\vartheta=\log x_1$$

on $\Omega:=\left\{z=(x,y)\in\mathbb{R}^{m+k}:x_1>1,y_1>1\right\}$ yields the following L^p Rellich type inequality with nonradial weights.

Corollary 3.5. *For any $u\in C_0^\infty(\Omega)$, one has*

$$\int_\Omega x_1^{2p-2}\log y_1\left|\Delta_\gamma u\right|^p dz\geq\int_\Omega\frac{|x|^{2\gamma}}{y_1^2\log^{p-1}x_1}|u|^p dz,$$

where $\Omega=\left\{z=(x,y)\in\mathbb{R}^{m+k}:x_1>1,y_1>1\right\}$.

4. Weighted Rellich type inequality II

In this section we prove the following sharp weighted L^p Rellich type inequality that connects first to second order derivatives. We should mention that the L^2 version of the following inequality has been obtained earlier in [12]. However, the previously employed technique does not give us the following form. The result given by the new technique also includes and improves (range of α is improved) the previously obtained result.

Theorem 4.1. *Let $Qp>Q+\alpha>p>1$ and $\alpha\in\mathbb{R}$. Then for all $u\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$, one has*

$$\int_{\mathbb{R}^n}\rho^\alpha\frac{\rho^{\gamma p}}{|x|^{\gamma p}}\left|\Delta_\gamma u\right|^p dz\geq\left(\frac{Qp-Q-\alpha}{p}\right)^p\int_{\mathbb{R}^n}\rho^{\alpha-p}\left|\nabla_\gamma u\right|^p dz. \tag{4.1}$$

Furthermore, the constant $(\frac{Qp-Q-\alpha}{p})^p$ is sharp.

Proof. Let $u\in C_0^\infty(\mathbb{R}^n\setminus\{0\})$ and define a new function

$$v=\rho^\beta\nabla_\gamma\rho\cdot\nabla_\gamma u, \tag{4.2}$$

where $\beta\in\mathbb{R}$. Noting the formulas (2.3) and (2.5), straightforward computation gives

$$\nabla_\gamma v=\rho^\beta\Delta_\gamma u\nabla_\gamma\rho+(Q+\beta-1)\rho^{\beta-1}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\nabla_\gamma u.$$

We now apply the following convexity inequality:

$$|\xi+\eta|^p\geq|\xi|^p+p|\xi|^{p-2}\xi\cdot\eta, \tag{4.3}$$

where $\xi, \eta \in \mathbb{R}^n$ and $p > 1$ (see [14]). This yields

$$\begin{aligned}\left|\rho^{\beta}\Delta_{\gamma}u\nabla_{\gamma}\rho\right|^{p} &= \left| -(Q+\beta-1)\,\rho^{\beta-1}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\nabla_{\gamma}u + \nabla_{\gamma}v\right|^{p} \\ &\geq (Q+\beta-1)^{p}\,\rho^{(\beta-1)p}\frac{|x|^{2\gamma p}}{\rho^{2\gamma p}}\left|\nabla_{\gamma}u\right|^{p} \\ &\quad - p\,(Q+\beta-1)^{p-1}\,\rho^{(\beta-1)(p-1)}\frac{|x|^{2\gamma(p-1)}}{\rho^{2\gamma(p-1)}}\left|\nabla_{\gamma}u\right|^{p-2}\nabla_{\gamma}u\cdot\nabla_{\gamma}v. \end{aligned} \tag{4.4}$$

It follows from (4.2) that $v\nabla_{\gamma}\rho\cdot\nabla_{\gamma}u = \rho^{\beta}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\left|\nabla_{\gamma}u\right|^{2}$ and hence

$$\left|\nabla_{\gamma}u\right|^{p-2}\nabla_{\gamma}u\cdot\nabla_{\gamma}v = \frac{\rho^{\gamma p}}{|x|^{\gamma p}}\frac{\rho^{(1-p)\beta}\nabla_{\gamma}\rho\cdot\nabla_{\gamma}\left|v\right|^{p}}{p}. \tag{4.5}$$

Combining (4.4) and (4.5), we obtain

$$\begin{aligned}\rho^{\beta p}\frac{|x|^{\gamma p}}{\rho^{\gamma p}}\left|\Delta_{\gamma}u\right|^{p} &\geq (Q+\beta-1)^{p}\,\rho^{(\beta-1)p}\frac{|x|^{2\gamma p}}{\rho^{2\gamma p}}\left|\nabla_{\gamma}u\right|^{p} \\ &\quad - (Q+\beta-1)^{p-1}\,\rho^{1-p}\frac{|x|^{\gamma(p-2)}}{\rho^{\gamma(p-2)}}\nabla_{\gamma}\rho\cdot\nabla_{\gamma}\left|v\right|^{p}. \end{aligned} \tag{4.6}$$

Taking $\beta = \frac{p-Q-\alpha}{p}$ and then multiplying both sides of (4.6) by $\frac{\rho^{2\gamma p+Q-p+2\alpha}}{|x|^{2\gamma p}}$, we have

$$\begin{aligned}\rho^{\alpha}\frac{\rho^{\gamma p}}{|x|^{\gamma p}}\left|\Delta_{\gamma}u\right|^{p} &\geq \ell^{p}(Q,p,\alpha)\rho^{\alpha-p}\left|\nabla_{\gamma}u\right|^{p} \\ &\quad - \ell^{p-1}(Q,p,\alpha)\frac{\rho^{\gamma(p+2)+Q-2p+2\alpha+1}}{|x|^{\gamma(p+2)}}\nabla_{\gamma}\rho\cdot\nabla_{\gamma}\left|v\right|^{p}, \end{aligned}$$

where $\ell(Q,p,\alpha) = \left(\frac{Qp-Q-\alpha}{p}\right)$. As an immediate consequence of integration by parts, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\rho^{\alpha}\frac{\rho^{\gamma p}}{|x|^{\gamma p}}\left|\Delta_{\gamma}u\right|^{p}dz &\geq \ell^{p}(Q,p,\alpha)\int_{\mathbb{R}^n}\rho^{\alpha-p}\left|\nabla_{\gamma}u\right|^{p}dz \\ &\quad + \ell^{p-1}(Q,p,\alpha)\int_{\mathbb{R}^n}\nabla_{\gamma}\cdot\left(\frac{\rho^{\gamma(p+2)+Q-2p+2\alpha+1}}{|x|^{\gamma(p+2)}}\nabla_{\gamma}\rho\right)\left|v\right|^{p}dz. \end{aligned}$$

Together with the identities (2.5) and (2.6), we compute

$$\nabla_{\gamma}\cdot\left(\frac{\rho^{\gamma(p+2)+Q-2p+2\alpha+1}}{|x|^{\gamma(p+2)}}\nabla_{\gamma}\rho\right) = 2\,(Q+\alpha-p)\,\frac{\rho^{\gamma p+Q+2\alpha-2p}}{|x|^{\gamma p}}.$$

Since $Qp > Q+\alpha > p > 1$, we obtain the desired inequality:

$$\int_{\mathbb{R}^n}\rho^{\alpha}\frac{\rho^{\gamma p}}{|x|^{\gamma p}}\left|\Delta_{\gamma}u\right|^{p}dz \geq \left(\frac{Qp-Q-\alpha}{p}\right)^{p}\int_{\mathbb{R}^n}\rho^{\alpha-p}\left|\nabla_{\gamma}u\right|^{p}dz.$$

To show that the constant $(\frac{Qp-Q-\alpha}{p})^{p}$ is sharp, we use the family of radial functions

$$u_{\epsilon}(\rho) = \begin{cases} -\left(\frac{Q+\alpha-2p}{p}+\epsilon\right)(\rho-1)+1 & \text{if} \quad \rho \leq 1, \\ \rho^{-\left(\frac{Q+\alpha-2p}{p}+\epsilon\right)} & \text{if} \quad \rho > 1, \end{cases}$$

and then pass to the limit as $\epsilon \longrightarrow 0$. □

5. Two-weight Rellich type inequalities with remainders

We now present a new method, which establishes two-weight L^p Rellich type inequalities with remainder terms. Our method relies on a two-weight Hardy type inequality with remainder terms proved in [13] and a nonlinear differential inequality including a nonnegative weight function a, the *gauge* function ρ and a positive smooth function δ.

Theorem 5.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Suppose $a \geq 0$ and $\delta > 0$ are smooth functions satisfying $-\nabla_\gamma \cdot (a\rho^{2-Q}\nabla_\gamma \delta) \geq 0$ almost everywhere in Ω. Then for all $u \in C_0^\infty(\Omega)$ the following inequality is valid:*

$$\begin{aligned}\int_\Omega a\rho^{\alpha+2p-2}\frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}}\,|\Delta_\gamma u|^p\,dz \geq c^p\int_\Omega a\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\,|u|^p\,dz\\ +(\frac{p-1}{p})c^{p-1}\int_\Omega a\rho^\alpha\frac{|\nabla_\gamma\delta|^2}{\delta^2}\,|u|^p\,dz\\ +\tfrac{2(p-1)(Q+\alpha-2)}{p}c^{p-1}\int_\Omega \rho^{\alpha-1}\nabla_\gamma\rho\cdot\nabla_\gamma a\,|u|^p\,dz\\ +2pc^{p-1}\int_\Omega\rho^\alpha\nabla_\gamma\rho\cdot\nabla_\gamma a\,|u|^{p-1}\,dz\\ +c^{p-1}\int_\Omega\rho^\alpha\Delta_\gamma a\,|u|^p\,dz,\end{aligned}\tag{5.1}$$

where $c = c\,(Q,p,\alpha) := (\frac{Q+\alpha-2}{p})(\frac{(Q-2)(p-1)-\alpha}{p})$ and $2 \leq p < \infty$, $Q-2 > |\alpha|$.

Proof. It follows from the formula (2.5) that

$$\Delta_\gamma\rho^\alpha = \alpha(Q+\alpha-2)\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}.\tag{5.2}$$

Multiplying both sides of (5.2) by the function $a\,|u|^p$ and applying the integration by parts formula two times over Ω, we have

$$\int_\Omega\rho^\alpha\Delta_\gamma\left(a\,|u|^p\right)dz = \alpha(Q+\alpha-2)\int_\Omega a\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\,|u|^p\,dz.\tag{5.3}$$

It is readily verified that

$$\begin{aligned}\Delta_\gamma\left(a\,|u|^p\right) = \Delta_\gamma a\,|u|^p + 2p\nabla_\gamma u\cdot\nabla_\gamma a\,|u|^{p-1}\\ +p\,(p-1)\,a\,|\nabla_\gamma u|^2\,|u|^{p-2} + pa\;\Delta_\gamma u\,|u|^{p-1}.\end{aligned}\tag{5.4}$$

By substituting (5.4) into the equality (5.3) and then rearranging the terms, we conclude that

$$\begin{aligned}\int_\Omega a\rho^\alpha\,|\nabla_\gamma u|^2\,|u|^{p-2}\,dz - \frac{\alpha(Q+\alpha-2)}{p\,(p-1)}\int_\Omega a\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\,|u|^p\,dz\\ =-\frac{1}{p-1}\int_\Omega a\rho^\alpha\Delta_\gamma u\,|u|^{p-1}\,dz - \frac{2}{p-1}\int_\Omega\rho^\alpha\nabla_\gamma a\cdot\nabla_\gamma u\,|u|^{p-1}\,dz\\ -\frac{1}{p\,(p-1)}\int_\Omega\rho^\alpha\Delta_\gamma a\,|u|^p\,dz.\end{aligned}\tag{5.5}$$

We now apply Hölder and Young inequalities to the integral expression

$$I_1 := -\tfrac{1}{p-1}\int_\Omega a\rho^\alpha\Delta_\gamma u\,|u|^{p-1}\,dz$$

on the right hand side of (5.5), which leads the following estimation:

$$I_1 \leq \frac{1}{p-1}\left(\int_\Omega a\frac{\rho^{\alpha+2(\gamma+1)(p-1)}}{|x|^{2\gamma(p-1)}}\,|\Delta_\gamma u|^p\,dz\right)^{\frac{1}{p}}\left(\int_\Omega a\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\,|u|^p\,dz\right)^{\frac{p-1}{p}}$$

$$\leq \frac{\epsilon^p}{p\,(p-1)}\int_\Omega a\frac{\rho^{\alpha+2(\gamma+1)(p-1)}}{|x|^{2\gamma(p-1)}}\,|\Delta_\gamma u|^p\,dz+\frac{\epsilon^{\frac{-p}{p-1}}}{p}\int_\Omega a\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\,|u|^p\,dz \tag{5.6}$$

for any $\epsilon>0$. As a next step, we first use the identity $|\nabla_\gamma u|^2\,|u|^{p-2}=\frac{4}{p^2}\left|\nabla_\gamma\left(u^{p/2}\right)\right|^2$ and then apply the L^2 version of the following improved weighted Hardy type inequality obtained in [13]:

$$\begin{aligned}\int_\Omega a\rho^\alpha|\nabla_\gamma u|^p dz \geq{}& \Big(\frac{Q+\alpha-p}{p}\Big)^p\int_\Omega a\rho^{\alpha-p}\frac{|x|^{\gamma p}}{\rho^{\gamma p}}\,|u|^p\,dz\\ &+\Big(\frac{Q+\alpha-p}{p}\Big)^{p-1}\int_\Omega \rho^{\alpha-p+1}\frac{|x|^{\gamma(p-2)}}{\rho^{\gamma(p-2)}}\nabla_\gamma\rho\cdot\nabla_\gamma a\,|u|^p\,dz\\ &+\frac{c_p}{p^p}\int_\Omega a\rho^\alpha\frac{|\nabla_\gamma\delta|^p}{\delta^p}\,|u|^p\,dz,\end{aligned} \tag{5.7}$$

to the integral expression $I_2:=\int_\Omega a\rho^\alpha\,|\nabla_\gamma u|^2\,|u|^{p-2}\,dz$ on the left hand side of the equality (5.5). We get

$$\begin{aligned}I_2=\frac{4}{p^2}\int_\Omega a\rho^\alpha\left|\nabla_\gamma\left(u^{p/2}\right)\right|^2dz \geq{}& \left(\frac{Q+\alpha-2}{p}\right)^2\int_\Omega a\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\,|u|^p\,dz\\ &+\frac{2\,(Q+\alpha-2)}{p^2}\int_\Omega\rho^{\alpha-1}\nabla_\gamma\rho\cdot\nabla_\gamma a\,|u|^p\,dz\\ &+\frac{1}{p^2}\int_\Omega a\rho^\alpha\frac{|\nabla_\gamma\delta|^2}{\delta^2}\,|u|^p\,dz.\end{aligned} \tag{5.8}$$

Plugging the inequalities (5.6) and (5.8) into the expression (5.5) and then rearranging the resulting inequality, we deduce that

$$\begin{aligned}\int_\Omega a\frac{\rho^{\alpha+2(\gamma+1)(p-1)}}{|x|^{2\gamma(p-1)}}\,|\Delta_\gamma u|^p\,dz \geq{}& f(Q,p,\alpha;\epsilon)\int_\Omega a\rho^{\alpha-2}\frac{|x|^{2\gamma}}{\rho^{2\gamma}}\,|u|^p\,dz\\ &+(\frac{p-1}{p\epsilon^p})\int_\Omega a\rho^\alpha\frac{|\nabla_\gamma\delta|^2}{\delta^2}\,|u|^p\,dz\\ &+\frac{2p}{\epsilon^p}\int_\Omega\rho^\alpha\nabla_\gamma a\cdot\nabla_\gamma u\,|u|^{p-1}\,dz\\ &+\frac{2\,(p-1)\,(Q+\alpha-2)}{p\epsilon^p}\int_\Omega\rho^{\alpha-1}\nabla_\gamma\rho\cdot\nabla_\gamma a\,|u|^p\,dz\\ &+\frac{1}{\epsilon^p}\int_\Omega\rho^\alpha\Delta_\gamma a\,|u|^p\,dz\end{aligned} \tag{5.9}$$

for any $\epsilon>0$ and

$$f(Q,p,\alpha;\epsilon):=\epsilon^{-p}\left[(\frac{p-1}{p})(Q+\alpha-2)^2-\alpha\,(Q+\alpha-2)-(p-1)\,\epsilon^{\frac{-p}{p-1}}\right].$$

Let us choose $\epsilon=c^{\frac{1-p}{p}}$ in the inequality (5.9), where $c=(\frac{Q+\alpha-2}{p})(\frac{(Q-2)(p-1)-\alpha}{p})$. This completes derivation of the desired result (5.1). □

Applications of Theorem 5.1. One can obtain as many two-weight L^p Rellich type inequalities with remainders as one can construct weight functions a and δ satisfying the hypotheses in the above theorem. For instance, consider the two functions

$$a \equiv 1 \quad \text{and} \quad \delta = \log(\frac{R}{\rho})$$

on a bounded domain Ω with smooth boundary in $\mathbb{R}^n$, where $R > \sup_{z\in\Omega} \rho$. It is obvious that they fulfill all assumptions in the Theorem 5.1. Hence, we have the weighted L^p Rellich type inequality including a logarithmic remainder.

Corollary 5.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Then for all $2 \le p < \infty$ and $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega \rho^{\alpha+2p-2} \frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} |\Delta_\gamma u|^p \, dz \ge c^p \int_\Omega \rho^{\alpha-2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p dz + (\frac{p-1}{p}) c^{p-1} \int_\Omega \frac{\rho^{\alpha-2}}{(\log \frac{R}{\rho})^2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p dz,$$

where $c = c\,(Q,p,\alpha) = (\frac{Q+\alpha-2}{p})(\frac{(Q-2)(p-1)-\alpha}{p})$ and $R > \sup_{z\in\Omega} \rho$, $Q - 2 > |\alpha|$.

By considering the pair as

$$a \equiv 1 \quad \text{and} \quad \delta = \log(\log \frac{R}{\rho}), \quad R > e \sup_{z\in\Omega} \rho,$$

we acquire the following result having a different logarithmic remainder.

Corollary 5.2. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Then for all $2 \le p < \infty$ and $u \in C_0^\infty(\Omega)$, there holds*

$$\int_\Omega \rho^{\alpha+2p-2} \frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} |\Delta_\gamma u|^p \, dz \ge c^p \int_\Omega \rho^{\alpha-2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p dz + (\frac{p-1}{p}) c^{p-1} \int_\Omega \frac{\rho^{\alpha-2}}{\left(\log \frac{R}{\rho} \log(\log \frac{R}{\rho})\right)^2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p dz.$$

Here $c = c\,(Q,p,\alpha) = (\frac{Q+\alpha-2}{p})(\frac{(Q-2)(p-1)-\alpha}{p})$ and $R > e \sup_{z\in\Omega} \rho$, $Q - 2 > |\alpha|$.

We now apply Theorem 5.1 with the pair

$$a = e^\rho \quad \text{and} \quad \delta = e^{-\rho};$$

then we get the subsequent two-weight L^p Rellich type inequality containing more nonnegative remainders.

Corollary 5.3. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Then for all $u \in C_0^\infty(\Omega)$, the following inequality is valid:*

$$\begin{aligned}\int_\Omega e^\rho \rho^{\alpha+2p-2} \frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} |\Delta_\gamma u|^p dz \geq c^p \int_\Omega e^\rho \rho^{\alpha-2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p \, dz \\ + (\frac{2p-1}{p}) c^{p-1} \int_\Omega e^\rho \rho^\alpha \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p \, dz \\ + (\tfrac{2(p-1)(Q+\alpha-2)+p(Q-1)}{p}) c^{p-1} \int_\Omega e^\rho \rho^{\alpha-1} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p \, dz \\ + 2pc^{p-1} \int_\Omega e^\rho \rho^\alpha \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^{p-1} \, dz,\end{aligned}$$

where $c = c(Q,p,\alpha) = (\frac{Q+\alpha-2}{p})(\frac{(Q-2)(p-1)-\alpha}{p})$ and $2 \leq p < \infty$, $Q - 2 > |\alpha|$.

Another consequence of Theorem 5.1 with the special functions

$$a \equiv 1 \quad \text{and} \quad \delta = R - \rho$$

on the $\rho-$ball B_R in $\mathbb{R}^n$ leads us to the following result.

Corollary 5.4. *Let B_R be the $\rho-$ball in $\mathbb{R}^n$ and $2 \leq p < \infty$, $Q - 2 > |\alpha|$. Then for every $u \in C_0^\infty(B_R)$, the following inequality is valid:*

$$\begin{aligned}\int_{B_R} \rho^{\alpha+2p-2} \frac{\rho^{2\gamma(p-1)}}{|x|^{2\gamma(p-1)}} |\Delta_\gamma u|^p \, dz \geq c^p \int_{B_R} \rho^{\alpha-2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p dz \\ + (\frac{p-1}{p}) c^{p-1} \int_{B_R} \frac{\rho^\alpha}{(R-\rho)^2} \frac{|x|^{2\gamma}}{\rho^{2\gamma}} |u|^p dz,\end{aligned}$$

where $c = c(Q,p,\alpha) = (\frac{Q+\alpha-2}{p})(\frac{(Q-2)(p-1)-\alpha}{p})$.

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References

[1] Mohamed Salah Baouendi, *Sur une classe d'opérateurs elliptiques dégénérés* (French), Bull. Soc. Math. France **95** (1967), 45–87. MR0228819

[2] G. Barbatis, *Improved Rellich inequalities for the polyharmonic operator*, Indiana Univ. Math. J. **55** (2006), no. 4, 1401–1422, DOI 10.1512/iumj.2006.55.2752. MR2269418

[3] G. Barbatis, *Best constants for higher-order Rellich inequalities in $L^p(\Omega)$*, Math. Z. **255** (2007), no. 4, 877–896, DOI 10.1007/s00209-006-0056-5. MR2274540

[4] Thomas Bieske, *Viscosity solutions on Grushin-type planes*, Illinois J. Math. **46** (2002), no. 3, 893–911. MR1951247

[5] Paolo Caldiroli and Roberta Musina, *Rellich inequalities with weights*, Calc. Var. Partial Differential Equations **45** (2012), no. 1-2, 147–164, DOI 10.1007/s00526-011-0454-3. MR2957654

[6] E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z. **227** (1998), no. 3, 511–523, DOI 10.1007/PL00004389. MR1612685

[7] Nicola Garofalo, *Unique continuation for a class of elliptic operators which degenerate on a manifold of arbitrary codimension*, J. Differential Equations **104** (1993), no. 1, 117–146, DOI 10.1006/jdeq.1993.1065. MR1224123

[8] Nassif Ghoussoub and Amir Moradifam, *Bessel pairs and optimal Hardy and Hardy-Rellich inequalities*, Math. Ann. **349** (2011), no. 1, 1–57, DOI 10.1007/s00208-010-0510-x. MR2753796

[9] V. V. Grušin, *A certain class of hypoelliptic operators* (Russian), Mat. Sb. (N.S.) **83 (125)** (1970), 456–473. MR0279436

[10] V. V. Grušin, *A certain class of elliptic pseudodifferential operators that are degenerate on a submanifold* (Russian), Mat. Sb. (N.S.) **84 (126)** (1971), 163–195. MR0283630

[11] Shou-feng Shen and Yong-yang Jin, *Rellich type inequalities related to Grushin type operator and Greiner type operator*, Appl. Math. J. Chinese Univ. Ser. B **27** (2012), no. 3, 353–362, DOI 10.1007/s11766-012-2908-6. MR2967336

[12] Ismail Kombe, *Hardy and Rellich-type inequalities with remainders for Baouendi-Grushin vector fields*, Houston J. Math. **41** (2015), no. 3, 849–874. MR3423688

[13] I. Kombe and A. Yener, *General Hardy type inequalities related to Baouendi-Grushin operators,* to appear in Complex Variables and Elliptic Equations.

[14] Peter Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. **109** (1990), no. 1, 157–164, DOI 10.2307/2048375. MR1007505

[15] Amir Moradifam, *Optimal weighted Hardy-Rellich inequalities on* $H^2 \cap H_0^1$, J. Lond. Math. Soc. (2) **85** (2012), no. 1, 22–40, DOI 10.1112/jlms/jdr045. MR2876309

[16] Giorgio Metafune, Motohiro Sobajima, and Chiara Spina, *Weighted Calderón-Zygmund and Rellich inequalities in* L^p, Math. Ann. **361** (2015), no. 1-2, 313–366, DOI 10.1007/s00208-014-1075-x. MR3302622

[17] Pengcheng Niu, Huiqing Zhang, and Yong Wang, *Hardy type and Rellich type inequalities on the Heisenberg group*, Proc. Amer. Math. Soc. **129** (2001), no. 12, 3623–3630, DOI 10.1090/S0002-9939-01-06011-7. MR1860496

[18] Franz Rellich, *Halbbeschränkte Differentialoperatoren höherer Ordnung* (German), Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956, pp. 243–250. MR0088624

[19] A. Tertikas and N. B. Zographopoulos, *Best constants in the Hardy-Rellich inequalities and related improvements*, Adv. Math. **209** (2007), no. 2, 407–459, DOI 10.1016/j.aim.2006.05.011. MR2296305

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