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Difference sequence spaces derived by using a generalized weighted mean

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ABSTRACT

In this work, we define new sequence spaces by combining a generalized weighted mean and a difference operator. Afterward, we investigate topological structures, which have completeness, *AK*-property, and *AD*-property. Also, we compute the α-, β- and γ-duals, and obtain bases for these sequence spaces. Finally, the necessary and sufficient conditions on an infinite matrix belonging to the classes $(c(u, v, \Delta) : \ell_\infty)$ and $(c(u, v, \Delta) : c)$ are obtained.

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1. Introduction

In the study on the sequence spaces, there are some basic approaches which are determination of topologies, matrix mapping and inclusions of sequence spaces (see; [1]). These methods are applied to study the matrix domain λ_A of an infinite matrix A defined by $\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}$. Although in the most cases the new sequence space λ_A generated by the limitation matrix A from a sequence space λ is the expansion or the contraction of the original space λ, in some cases it may be observed that those spaces overlap. Indeed, one can easily see that the inclusions $\lambda_S \subset \lambda$ and $\lambda \subset \lambda_\Delta$ strictly hold for $\lambda \in \{\ell_\infty, c, c_0\}$ (see; [2]). Especially, the generalized weighted mean and the difference operator which are special cases for the matrix A have been studied extensively via the methods mentioned above.

In the literature, some new sequence spaces are defined by using the generalized weighted mean and the difference operator or by combining both of them. For example, in [3], the difference sequence spaces are first defined by Kızmaz. Further, the authors including Ahmad and Mursaleen [4], Çolak and Et [5], Başar and Altay [6], Karakaya and Polat [7], and the others have defined and studied new sequence spaces by considering matrices that represent difference operators. The articles concerning this work can be found in the list of Refs. [8–11]. On the other hand, by using a generalized weighted mean, several authors defined some new sequence spaces and studied some properties of these spaces. Some of them are as follows: Malkowsky and Savaş [12] have defined the sequence spaces $Z(u, v, \lambda)$ which consists of all sequences such that $G(u, v)$-transforms of them are in $\lambda \in \{\ell_\infty, c, c_0, \ell_p\}$. Başar and Altay [13–15] have defined and studied the sequence spaces of nonabsolute type derived by using the generalized weighted mean over the paranormed spaces.

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In this work, our purpose is to introduce new sequence spaces by combining the generalized weighted mean and the difference operator and also investigate topological structures, which have completeness, *AK*-property and *AD*-property, the α-, β-, γ- duals, and the bases of these sequence spaces. In addition, we characterize some matrix mappings on these spaces.

2. Preliminaries and notations

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. We write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1, we denote the spaces of all bounded, convergent and absolutely convergent series, respectively.

A sequence space λ with a linear topology is called a *K*-space provided each of the maps $p_i : \lambda \to C$ defined by $p_i(x) = x_i$ is continuous for all $i \in N$; where C denotes the complex field and $N = \{0, 1, 2, \ldots\}$. A *K*-space λ is called an *FK* space provided λ is a complete linear metric space. An *FK*-space whose topology is normable is called a *BK*-space. An *FK*-space λ is said to have *AK*-property, if $\varphi \subset \lambda$ and $\{e^{(k)}\}$ is a basis for λ, where $e^{(k)}$ is a sequence whose only non-zero term is 1 in kth place for each $k \in N$ and $\varphi = \text{span}\{e^{(k)}\}$, the set of all finitely non-zero sequences. If φ is dense in λ, then λ is called an *AD*-space, thus *AK* implies *AD*. For example, the spaces c_0, cs, and ℓ_p are *AK*-spaces, where $1 < p < \infty$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in N$. Then, we write $Ax = ((Ax)n)$, the A-transform of x, if $A_n(x) = \sum_k a_{nk}x_k$ converges for each $n \in N$. If $x \in \lambda$ implies that $Ax \in \mu$, then we say that A defines a matrix mapping from λ into μ and denote it by $A : \lambda \to \mu$. By $(\lambda : \mu)$, we mean the class of all infinite matrices A such that $A : \lambda \to \mu$. Also we denote all finite subsets of N by F. We write $e = (1, 1, 1, \ldots)$ and U for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in N$. For $u \in U$, let $1/u = (1/u_n)$. Let $u, v \in U$ and let us define the matrix $G(u, v) = (g_{nk})$ as

$$g_{nk} = \begin{cases} u_n v_k & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

for all $k, n \in N$, where u_n depends only on n and v_k only on k. The matrix $G(u, v)$, defined above, is called as generalized weighted mean or factorable matrix.

The *continuous dual* X' of a normed space X is defined as the space of all bounded linear functionals on X. If A is triangle, that is $a_{nk} = 0$ if $k > n$ and $a_{nn} \neq 0$ for all $n \in N$, and λ is a sequence space, then $f \in \lambda'_A$ if and only if $f = g \circ A$, $g \in \lambda'$.

Let X be a seminormed space. A set $Y \subset X$ is called fundamental if the span of Y is dense in X. One of the useful results on the fundamental set which is an application of the Hahn–Banach theorem is as follows: if Y is the subset of a seminormed space X and $f \in X'$, $f(Y) = 0$ implies $f = 0$, then Y is fundamental [16, p. 39].

3. The sequence spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$

In this section, we define the new sequence spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ derived by using the generalized weighted mean, and prove that these are the complete normed linear spaces and compute their α-, β-, and γ- duals. Furthermore, we give the basis for the spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{c, c_0\}$. Also we show that these spaces show *AK* and *AD* properties.

We define the sequence spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ by

$$\lambda(u, v, \Delta) = \left\{ x = (x_k) \in w : (y_k) = \left(\sum_{i=1}^{k} u_k v_i \Delta x_i \right) \in \lambda \right\}.$$

We write $\Delta x = (\Delta x_k)$ for the sequence $(x_k - x_{k-1})$ and use the convention that any term with a negative subscript is equal to naught.

If λ is any normed sequence space, then we call the matrix domain $\lambda_{G(u,v,\Delta)}$ as the generalized weighted mean and the difference sequence space. It is natural that these spaces may also be defined according to the matrix domain as follows:

$$\lambda(u, v, \Delta) = \lambda_{G(u,v,\Delta)}$$

where $G(u.v, \Delta) = G(u, v).\Delta$.

Define the sequence $y = (y_k)$, which will be frequently used as the $G(u.v, \Delta)$-transform of a sequence $x = (x_k)$ i.e.,

$$y_k = \sum_{i=0}^{k} u_k v_i \Delta x_i = \sum_{i=0}^{k} u_k \nabla v_i x_i, \ (\nabla v = v_i - v_{i+1}) \quad (k \in N). \tag{3.1}$$

Since the proof may also be obtained in the similar way as for the other spaces, to avoid the repetition of the similar statements, we give the proof only for one of those spaces.

Theorem 1. *The sequence spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ are complete normed linear spaces with respect to the norm defined by*

$$\|x\|_{\lambda(u,v,\Delta)} = \sup_k \left| \sum_{i=0}^{k} u_k v_i \Delta x_i \right| = \|y\|_\lambda. \tag{3.2}$$

Proof. The linearity of $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ with respect to the coordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x, t \in \lambda(u, v, \Delta)$ with $\lambda \in \{\ell_\infty, c, c_0\}$ and $\alpha, \beta \in R$

$$\sup_{k\in N}\left|\sum_{i=0}^{k} u_k v_i \Delta\left(\alpha x_i + \beta t_i\right)\right| \leq |\alpha| \sup_{k\in N}\left|\sum_{i=0}^{k} u_k v_i \Delta x_i\right| + |\beta| \sup_{k\in N}\left|\sum_{i=0}^{k} u_k v_i \Delta t_i\right|. \tag{3.3}$$

After this step, we must show that the spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ hold the norm conditions and the completeness with respect to the given norm. It is easy to show that (3.2) holds the norm condition for the spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$. To prove the completeness of the space $\ell_\infty(u, v, \Delta)$, let us take any Cauchy sequence (x^n) in the space $\ell_\infty(u, v, \Delta)$. Then for a given $\varepsilon > 0$, there exists a positive integer $N_0(\varepsilon)$ such that $\|x^n - x^r\|_{\lambda(u,v,\Delta)} < \varepsilon$ for all $n, r > N_0(\varepsilon)$. Hence for fixed $i \in N$,

$$\left|G(u.v, \Delta)(x_i^n - x_i^r)\right| < \varepsilon$$

for all $n, r \geq N_0(\varepsilon)$. Therefore the sequence $((G\Delta)x^n)$ is a Cauchy sequence of real numbers for every $n \in N$. Since R is complete, it converges, that is,

$$((G(u.v, \Delta))x^r)_{i\in N} \to ((G(u.v, \Delta))x)_{i\in N}$$

as $r \to \infty$. So we have

$$\left|G(u.v, \Delta)(x_i^n - x_i)\right| < \varepsilon$$

for every $n \geq N_0(\varepsilon)$ and as $r \to \infty$. This implies that $\|x^n - x\|_{\lambda(u,v,\Delta)} < \varepsilon$ for every $n \geq N_0(\varepsilon)$. Now we must show that $x \in \ell_\infty(u, v, \Delta)$. We have

$$\sup_k |(G(u.v, \Delta)x)_k| \leq \left\|x^n\right\|_{\lambda(u,v,\Delta)} + \left\|x^n - x\right\|_{\lambda(u,v,\Delta)} = O(1).$$

This implies that $x = (x_i) \in \ell_\infty(u, v, \Delta)$. Therefore $\ell_\infty(u, v, \Delta)$ is a Banach space. It can be shown that $c(u, v, \Delta)$ and $c_0(u, v, \Delta)$ are closed subspaces of $\ell_\infty(u, v, \Delta)$ which lead us to the consequence that the spaces $c(u, v, \Delta)$ and $c_0(u, v, \Delta)$ are also Banach spaces with the norm (3.2).

Furthermore, since $\ell_\infty(u, v, \Delta)$ is a Banach space with continuous coordinates, i.e.,
$\left\|G(u.v, \Delta)(x^k - x)\right\|_{\lambda(u,v,\Delta)} \to 0$ implies $\left|G(u.v, \Delta)(x_i^k - x_i)\right| \to 0$ for all $i \in N$. Therefore, it is a *BK*-space. □

Theorem 2. *The sequence spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ are linearly isomorphic to the spaces $\lambda \in \{\ell_\infty, c, c_0\}$, respectively, i.e., $\ell_\infty(u, v, \Delta) \cong \ell_\infty$, $c(u, v, \Delta) \cong c$ and $c_0(u, v, \Delta) \cong c_0$.*

Proof. To prove the fact $c_0(u, v, \Delta) \cong c_0$, we should show the existence of a linear bijection between the spaces $c_0(u, v, \Delta)$ and c_0. Consider the transformation T defined with the Eq. (3.1), from $c_0(u, v, \Delta)$ to c_0 by $x \to y = Tx$. The linearity of T is clear.

Further, it is trivial that $x = 0$ whenever $Tx = 0$, and hence T is injective.

Let $y \in c_0$ and let us define the sequence $x = \{x_k\}$ as

$$x_k = \sum_{i=0}^{k-1} \frac{1}{u_k}\left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) y_i + \frac{1}{u_k v_k} y_k \quad (k \in N). \tag{3.4}$$

Then

$$\lim_{k\to\infty} (G(u.v, \Delta)x)_k = \lim_{k\to\infty} \sum_{i=0}^{k} u_k v_k \Delta x_i = \lim_{k\to\infty} y_k = 0.$$

Thus we have $x \in c_0(u, v, \Delta)$. Consequently, T is surjective and is norm preserving. Hence, T is a linear bijection which therefore says that the spaces $c_0(u, v, \Delta)$ and c_0 are linearly isomorphic. In the same way, it can be shown that $c(u, v, \Delta)$ and $\ell_\infty(u, v, \Delta)$ are linearly isomorphic to c and ℓ_∞, respectively, and so, we omit the details. □

Now we define a Schauder basis of a normed space. If a normed sequence space λ contains a sequence (b_n) such that, for every $x \in \lambda$, there is unique sequence of scalars (α_n) for which

$$g\left(x - \sum_{k=0}^{n} \alpha_n b_k\right) \to 0 \quad \text{as } n \to \infty.$$

Then (b_n) is called a Schauder basis for λ. The series $\sum \alpha_k b_k$ that has the sum x is called the expansion of x in (b_n), and we write $x = \sum \alpha_k b_k$, Maddox [17, p. 98].

By using the isomorphism T defined in Theorem 2, it can be shown that the inverse image of the basis of spaces c_0 and c is onto, and so they are the basis of the new spaces $c_0(u, v, \Delta)$ and $c(u, v, \Delta)$, respectively. Therefore, we give the following theorem without proof:

Theorem 3. *Let $\lambda_k = (G(u.v, \Delta)x)_k$ for all $k \in N$. We define the sequence $b^{(k)} = \{b_n^{(k)}\}_{n\in N}$ of the elements of the space $c_0(u, v, \Delta)$ as*

$$b_n^{(k)} = \begin{cases} \dfrac{1}{u_n v_k} - \dfrac{1}{u_n v_{k+1}} & (0 < k < n), \\ \dfrac{1}{u_n v_n} & (k = n), \\ 0 & (k > n) \end{cases}$$

for every fixed $k \in N$. Then the following assertions are true:

(i) *The sequence $\{b^{(k)}\}_{k\in N}$ is a basis for the space $c_0(u, v, \Delta)$, and any $x \in c_0(u, v, \Delta)$ has a unique representation in the form*

$$x = \sum_k \lambda_k b^{(k)}.$$

(ii) *The set $\{e, b^{(k)}\}$ is a basis for the space $c(u, v, \Delta)$, and any $x \in c(u, v, \Delta)$ has a unique representation in the form*

$$x = le + \sum_k (\lambda_k - l)b^{(k)},$$

where $\lambda_k = (G(u.v, \Delta)x)_k$ for all $k \in N$ and $l = \lim_{k\to\infty}(G(u.v, \Delta)x)_k$.

Theorem 4. *The sequence space $c_0(u, v, \Delta)$ has AD-property whenever $u \in c_0(u, v, \Delta)$.*

Proof. Suppose that $f \in [c_0(u, v, \Delta)]'$. Then there exists a functional g over the space c_0 such that $f(x) = g(G(u.v, \Delta)x)$ for some $g \in c_0' = \ell_1$. Since c_0 has *AK-property* and $c_0' \cong \ell_1$

$$f(x) = \sum_{j=1}^{\infty} a_j \sum_{i=1}^{j} u_j \nabla v_i x_i$$

for some $a = (a_j) \in \ell_1$. Since $u \in c_0$ by hypothesis, the inclusion $\varphi \subset c_0(u, v, \Delta)$ holds. For any $f \in [c_0(u, v, \Delta)]'$ and $e^{(k)} \in \varphi \subset c_0(u, v, \Delta)$, we have

$$f(e^{(k)}) = \sum_{j=1}^{\infty} a_j \left\{G(u.v, \Delta)e^{(k)}\right\}_j = \left\{G'(u.v, \Delta)a\right\}_k$$

where $G'(u.v, \Delta)$ is the transpose of the matrix $G(u.v, \Delta)$. Hence, from Hahn–Banach theorem, φ is dense in $c_0(u, v, \Delta)$ if and only if $G'(u.v, \Delta)a = \theta$ for $a \in \ell_1$ implies $a = \theta$. Since the null space of the operator $G'(u.v, \Delta)$ on w is $\{\theta\}$, $c_0(u, v, \Delta)$ has *AD*-property. □

We now give the details about duals of the sequence spaces $\lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$.

For the sequence spaces λ and μ, we define the set $S(\lambda, \mu)$ as

$$S(\lambda, \mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x \in \lambda\}. \tag{3.5}$$

With the notation of (3.5), the α-, β-, and γ- duals of a sequence space λ, which are denoted respectively by λ^α, λ^β, and λ^γ are defined in [18] as:

$$\lambda^\alpha = S(\lambda, l_1), \qquad \lambda^\beta = S(\lambda, cs) \quad \text{and} \quad \lambda^\gamma = S(\lambda, bs).$$

We now need the following lemmas due to Stieglitz and Tietz [19] for next theorems.

Lemma 1. *$A \in (c_0 : l_1)$ if and only if*

$$\sup_{K\in F} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty.$$

Lemma 2. *$A \in (c_0 : c)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty,$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0.$$

Lemma 3. *$A \in (c_0 : \ell_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty.$$

Theorem 5. *Let $u, v \in U$, $a = (a_k) \in w$ and we define the matrix $B = (b_{nk})$ as*

$$b_{nk} = \begin{cases} \left(\dfrac{1}{u_n v_k} - \dfrac{1}{u_n v_{k+1}}\right) a_k & (0 \le k \le n), \\ \dfrac{1}{u_n v_n} a_n & (k = n), \\ 0 & (k > n) \end{cases}$$

for all $k, n \in N$. Then the α-dual of the space $\lambda(u, v; \Delta)$ is the set

$$b_\Delta = \left\{ a = (a_k) \in w : \sup_{K \in F} \sum_n \left| \sum_{k \in K} \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_k + \frac{1}{u_k v_k} a_k \right| < \infty \right\}.$$

Proof. Let $a = (a_k) \in w$ and consider the matrix $G^{-1}(u.v, \Delta) = \Delta^{-1} G^{-1}(u, v)$ and sequence $a = (a_k)$. Bearing in mind the relation (3.1), we immediately derive that

$$\begin{aligned} a_k x_k &= \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) y_i a_k + \frac{1}{u_k v_k} y_k a_k \\ &= \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_i y_k + \frac{1}{u_k v_k} y_k a_k = (By)_k \end{aligned} \tag{3.6}$$

for all $i, k \in N$. We therefore observe by (3.6) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in \lambda(u, v, \Delta)$ for $\lambda \in \{\ell_\infty, c, c_0\}$ if and only if $By \in \ell_1$ whenever $y \in \lambda$, where $\{\ell_\infty, c, c_0\}$. Then, we derive by Lemma 1 that

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_k + \frac{1}{u_k v_k} a_k \right| < \infty$$

which yields the consequence that $[c_0(u, v, \Delta)]^\alpha = [c(u, v, \Delta)]^\alpha = [\ell_\infty(u, v, \Delta)]^\alpha = b_\Delta$. □

Theorem 6. *Let $u, v \in U$, $a = (a_k) \in w$ and the matrix $C = (c_{nk})$ by*

$$c_{nk} = \begin{cases} \left(\dfrac{1}{u_n v_k} - \dfrac{1}{u_n v_{k+1}}\right) a_k & (0 \le k < n), \\ \dfrac{1}{u_n v_n} a_n & (k = n), \\ 0 & (k > n) \end{cases}$$

and define the sets c_1, c_2, c_3, c_4 by

$$c_1 = \left\{ a = (a_k) \in w : \sup_n \sum_n |c_{nk}| < \infty \right\}; \qquad c_2 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} c_{nk} \text{ exists for each } k \in N \right\};$$

$$c_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k |c_{nk}| = \sum_k \left| \lim_{n \to \infty} c_{nk} \right| \right\}; \qquad c_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_k c_{nk} \text{ exists} \right\}.$$

Then, $[c_0(u, v, \Delta)]^\beta$, $[c(u, v, \Delta)]^\beta$ and $[\ell_\infty(u, v, \Delta)]^\beta$ are the sets $c_1 \cap c_2$, $c_1 \cap c_2 \cap c_4$ and $c_2 \cap c_3$ respectively.

Proof. Since the proof may be obtained in the similar way as for the spaces $c(u, v, \Delta)$ and $\ell_\infty(u, v, \Delta)$, we give only the proof for the space $c_0(u, v, \Delta)$ here. Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) y_i + \frac{1}{u_k v_k} y_k \right] a_k \\ &= \sum_{i=0}^{n} \left[\sum_{i=0}^{n} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}}\right) a_i + \frac{1}{u_k v_k} a_k \right] y_k \\ &= (Cy)_n. \end{aligned} \tag{3.7}$$

Thus, we deduce from Lemma 2 and (3.7) that $ax = (a_n x_n) \in cs$ whenever $x \in c_0(u, v, \Delta)$ if and only if $Cy \in c$ whenever $y \in c_0$. Therefore we derive by Lemma 2 which shows that $\{c_0(u, v, \Delta)\}^\beta = c_1 \cap c_2$. □

Theorem 7. *The γ-dual of $\lambda(u, v, \Delta)$ is the set c_1, where $\lambda \in \{\ell_\infty, c, c_0\}$.*

Proof. This may be obtained in the similar way, as mentioned in the proof of Theorem 6 with Lemma 3 instead of Lemma 2. So, we omit the details. □

4. Matrix transformations on the space $c(u, v, \Delta)$

In this section, we directly prove the theorems which characterize the classes $(c(u, v, \Delta) : \ell_\infty)$ and $A \in (c(u, v, \Delta) : c)$.

Theorem 8. *$A \in (c(u, v, \Delta) : \ell_\infty)$ if and only if*

$$\sup_n \sum_{k=0}^{n} \left| \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right| < \infty \tag{4.1}$$

$$\lim_{n\to\infty} \left[\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right] \tag{4.2}$$

exists for all $k, n \in N$.

$$\sup_{n\in N} \sum_{k=0}^{n} \left| \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right| < \infty, \quad (n \in N). \tag{4.3}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left[\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right] \tag{4.4}$$

exists for all $n \in N$.

Proof. Let $A \in (c(u, v, \Delta) : \ell_\infty)$. Then Ax exists and is in ℓ_∞ for all $x \in c(u, v, \Delta)$. Thus, since $\{a_{nk}\}_{k\in N} \in \{c(u, v, \Delta)\}^\beta$ for all $n \in N$, and since Ax exists and is in ℓ_∞ for every $x \in c(u, v, \Delta)$, the necessities of the conditions (4.2)–(4.4) are clear.

Let us consider the equality

$$\sum_{k=0}^{n} a_{nk} x_k = \sum_{k=0}^{n} \left[\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) + \frac{1}{u_k v_k} \right] a_{nk} y_k; \quad (n \in N)$$

which yields us under the assumption that as $n \to \infty$,

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right] y_k, \quad (n \in N). \tag{4.5}$$

Therefore we get by (4.5) that

$$\begin{aligned} \|Ax\|_\infty &\le \sup_n \sum_k \left| \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right| |y_k| \\ &\le \|y\|_\infty \sup_n \sum_k \left| \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right| < \infty. \end{aligned}$$

This means that $Ax \in \ell_\infty$ whenever $x \in c(u, v, \Delta)$ and this step completes the proof. □

We give a lemma concerning the characterization of the class $A \in (c_0 : c)$ which is needed in proving the next theorem and due to Stieglitz and Tietz [19].

Lemma 4. *$A \in (c_0 : c)$ if and only if it holds the conditions of Lemma 2 with $\alpha_k = 0$ for all k, and*

$$\lim_{n\to\infty} \sum_k a_{nk} = 0.$$

Theorem 9. *$A \in (c(u, v, \Delta) : c)$ if and only if (4.1)–(4.4) hold, and*

$$\lim_{n\to\infty} \sum_k \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} = \alpha, \tag{4.6}$$

$$\lim_{n\to\infty} \left(\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right) = \alpha_k, \quad (k \in N). \tag{4.7}$$

Proof. Let $A \in (c(u, v, \Delta) : c)$. Since $c \subset \ell_\infty$, the necessities of (4.1)–(4.4) are immediately obtained in Theorem 8. Since Ax exists and is in c for all $x \in c(u, v, \Delta)$ by the hypothesis, the necessities of the conditions (4.6) and (4.7) are easily obtained with the sequences $x = (1, 2, 3, \ldots)$ and $x = b^{(i)}$, respectively; where $b^{(i)}$ is defined by (3.2). Conversely, suppose that the conditions (4.1)–(4.4), (4.6) and (4.7) hold and take any $x \in c(u, v, \Delta)$. Then $\{a_{ni}\}_{i\in N} \in \{c(u, v, \Delta)\}^\beta$ for each $n \in N$ which implies that Ax exists. One can derive by (4.1) and (4.7) that

$$\sum_{k=0}^{n} |\alpha_k| \le \sup_n \sum_{k=0}^{n} \left| \sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right| < \infty$$

holds for every $n \in N$. This yields that $(\alpha_i) \in \ell_1$ and hence the series $\sum \alpha_i y_i$ absolutely converges. Let us consider the following equality obtained from Lemma 3 with $a_{ni} - \alpha_i$ instead of a_{ni}

$$\sum_k (a_{nk} - \alpha_k) x_k = \sum_k \left[\sum_{i=0}^{k-1} \frac{1}{u_k} \left(\frac{1}{v_i} - \frac{1}{v_{i+1}} \right) a_{nk} + \frac{1}{u_k v_k} a_{nk} \right] (a_{nk} - \alpha_i) y_k. \tag{4.8}$$

Therefore we derive from Lemma 2 with (4.8) that

$$\lim_{n\to\infty} \sum_k (a_{nk} - \alpha_k) x_k = 0. \tag{4.9}$$

Thus, we deduce by combining (4.9) with the fact $(\alpha_i y_i) \in \ell_1$ that $Ax \in c$ and this step completes the proof. □

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