Research Article

Necip Şimşek and Murat Kirişci*

Incomplete Fermatean fuzzy preference relations and group decision-making

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Abstract: There may be cases where experts do not have in-depth knowledge of the problem to be solved in decision-making problems. In such cases, experts may fail to express their views on certain aspects of the problem, resulting in incomplete preferences, in which some preference values are not provided or are missing. In this article, we present a new model for group decision-making (GDM) methods in which experts' preferences can be expressed as incomplete Fermatean fuzzy preference relations. This model is guided by the additive-consistency property and only uses the preference values the expert provides. An additive consistency definition characterized by a Fermatean fuzzy priority vector has been given. The additive consistency property is also used to measure the level of consistency of the information provided by the experts. The proposed additive consistency definition's property is presented, as well as a model for obtaining missing judgments in incomplete Fermatean fuzzy preference relations. We present a method for adjusting the inconsistency for Fermatean fuzzy preference relations, a model for obtaining the priority vector, and a method for increasing the consensus degrees of Fermatean fuzzy preference relations. In addition, we present a GDM method in environments with incomplete Fermatean fuzzy preference relations. To show that our method outperforms existing GDM methods in incomplete Fermatean fuzzy preference relations environments, we have provided an example and compared it with some methods. It has been seen that our proposed GDM method is beneficial for GDM in deficient Fermatean fuzzy preference relation environments and produces meaningful results for us.

Keywords: Fermatean fuzzy set, incomplete preference relations, additive consistency, decision-making

MSC 2020: 03E72, 68W01, 49N30

1 Introduction

1.1 Decision-making

In general, decision-making can be defined as selecting the best alternative from a set of multiple options based on the objectives and criteria established. A decision problem consists of the decision maker, analyst, decision variables, objectives, criteria, and constraints. A mathematical decision model is established for the evaluation and selection of behavioral alternatives. One or more objectives and criteria to be determined by the decision maker comprise evaluation measures.

When evaluating more than one criterion in a decision problem, such decision-making situations are referred to as multicriteria decision-making (MCDM) problems. MCDM techniques, which were developed to

* **Corresponding author: Murat Kirişci,** Department of Biostatistics and Medical Informatics, Istanbul University-Cerrahpasa, 34452, Istanbul, Turkey, e-mail: mkirisci@hotmail.com
Necip Şimşek: Department of Mathematics, Istanbul Commerce University, 34445, Istanbul, Turkey, e-mail: necipsimsek@hotmail.com

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solve MCDM problems, are used in decision-making when there are many and frequently incompatible criteria. Given the complexity of the problems encountered in practice, MCDM techniques provide significant benefits that facilitate correct decision-making by allowing the evaluation of many criteria and alternatives together and simultaneously.

Natural language information, which is commonly used by people, is ignored in the traditional decision-making process. However, decisions made in the presence of verbal ambiguity expressions must result in consistent and correct decisions. Decision-making becomes difficult, especially when there is insufficient or ambiguous information and factors containing verbal and linguistic uncertainties are frequently used. These elements suggest that the decision-making process should take place in a hazy environment.

Individuals want to overcome the challenges they face to achieve their objectives. This task can sometimes become so large and complex that the individual is unable to complete it alone. In such cases, using group power to make decisions is a more rational approach. The synergy that emerges as a group, whether working around a desk or dispersed in digital environments, is an important tool in improving decisions and solving problems. As a result, individuals achieve some of their needs and goals that they would not be able to achieve alone through groups.

Individuals understand problems better than groups. Individuals are accountable for the decisions in which they participate. Individuals are better at catching mistakes than groups. The group's information is the sum of the information of all participants. As a result, there are many alternatives for solving any problem, and far more solutions can be developed than through personal work. People and processes can benefit from the group work. The risk is reduced to the bare minimum. The group normalizes high-risk individuals and encourages dialogue. Subjectivity is less of a risk. People eliminate this risk by warning one another. Many different ideas are generated. Although it varies depending on the structure of the group and the subject, the group generates more ideas than the individual. People, with a few exceptions, influence one another, and rational ideas emerge through associations. It hastened the decision's implementation. Because personal responsibility is felt for the decision, it is also defended individually.

Although the belief that group decisions are superior to individual decisions, in general, is prevalent, it is also true that obtaining the benefits and advantages expected from group decisions is not always easy. It would be more realistic to discuss only the superiority of an individual or group decision based on the circumstances.

1.2 Preference relations

In the process of decision-making, a decision-maker is usually asked to give his or her preferences over alternatives. Preference relations (or called pairwise comparison matrices or judgment matrices) are very useful in expressing decision makers' preferences in decision problems in various fields, such as politics, social psychology, engineering, management, business, economics, etc.

Traditionally, a preference relation on a nonempty choice set X is defined as a complete, reflexive, and transitive binary relation (that is, as a complete preorder) on X. All of these assumptions are commonly viewed as rationality postulates, the latter two being consistency requirements and the former being a decisiveness prerequisite.

A complete preference of order n necessitates the completion of all $n(n-1)/2$ judgments in the top triangular portion of the triangle. Sometimes, however, it is difficult to obtain such a preference relation, especially for the preference relation of a high order, because of time pressure, lack of knowledge, and the decision maker's limited expertise related to the domain, the decision maker may develop an incomplete preference relation in which some of the elements cannot be provided.

One of these properties is the so-called consistency property. The lack of consistency in decision-making can lead to inconsistent conclusions; that is why it is important, if not crucial, to study conditions under which consistency is satisfied [17,34]. On the other hand, perfect consistency is difficult to obtain in practice, especially when measuring preferences on a set with a large number of alternatives.

The problem of consistency itself includes two problems [23]: (i) when an expert, considered individually, is said to be consistent; and (ii) when a whole group of experts is considered consistent.

The concept of consistency has traditionally been defined in terms of acyclicity [35], that is, the absence of sequences such as $x_1, x_2, \ldots, x_k (x_{k+1} = x_1)$ within a crisp model, where an expert provides his/her opinion on the set of alternatives, $X = x_1, x_2, \ldots, x_n$; $n \geq 2$, via a binary preference relation, R.

In a fuzzy context, where an expert expresses his/her opinions using fuzzy preference relations, a traditional requirement to characterize consistency is using transitivity, in the sense that if an alternative x_i is preferred to alternative x_j and this one to x_k, then alternative x_i should be preferred to x_k. Stronger conditions have been given to defining consistency, for example, max-min transitivity property or additive transitivity property [17,40,41]. However, the problem is the difficulty of checking and guaranteeing such consistent conditions in the decision-making processes.

The preference relation is the most common representation of information used in decision-making problems because it is a useful tool in modeling decision processes, above all when we want to aggregate experts' preferences into group preferences [17,34]. In a preference relation, an expert associates every pair of alternatives with a value that reflects some degree of preference for the first alternative over the second one. Many important decision models have been developed using mainly two kinds of preference relations: multiplicative preference relations and fuzzy preference relations.

In a preference relation, an expert associates to each pair of alternatives a real number that reflects the preference degree, or the ratio of preference intensity, of the first alternative over, or to that of, the second one. Two questions immediately arise when doing this.

- Which scale should be used to associate preference values with judgments?
- Which conditions have to be satisfied to obtain consistent results?

The answer to the first question depends on the selected model we are working with. As we mentioned earlier, the best-known choice models are multiplicative preference relations and Fuzzy preference relations.

(i) Fuzzy model: In this case, preferences are represented by a fuzzy preference relation P on a set of alternatives X, i.e., a fuzzy set on the product set $X \times X$, which is characterized by its membership function $\zeta_P : X \times X \to [0, 1]$. This implies that the scale to use in the fuzzy model is the closed interval $[0, 1]$.

(ii) Multiplicative model: In this case, preferences are represented using a multiplicative preference relation, $A = (a_{ij})$, on a set of alternatives X, being a_{ij} interpreted as the ratio of the preference intensity of alternative x_i to that of x_j. Saaty suggests measuring a_{ij} using a ratio scale, and precisely the $1 - 9$ scale [11], or more generally the closed interval $[1/9, 9]$.

For making a consistent choice when assuming fuzzy preference relations, a set of consistency properties to be satisfied by such relations has been suggested. Transitivity is one of the most important properties concerning preferences, and it represents the idea that the preference value obtained by comparing directly two alternatives should be equal to or greater than the preference value between those two alternatives obtained using an indirect chain of alternatives [5,9,15]. Some of the suggested properties are triangle condition, weak transitivity, max-min transitivity, max-max transitivity, restricted max-min transitivity, restricted max-max transitivity, multiplicative transitivity, and additive transitivity.

1.3 Motivation

Group decision-making (GDM) consists of multiple individuals interacting to reach a decision. Each decision maker (expert) may have unique motivations or goals and may approach the decision process from a different angle, but have a common interest in reaching an eventual agreement on selecting the "best" option(s) [32]. To do this, experts have to express their preferences using a set of evaluations over a set of

alternatives. It has been a common practice in research to model GDM problems in which all the experts express their preferences using the same preference representation format. However, in real practice, this is not always possible because each expert has unique characteristics in knowledge, skills, experience, and personality, which implies that different experts may express their evaluations using different preference representation formats. In fact, this is an issue that recently has attracted the attention of many researchers in the area of GDM, and as a result, different approaches to integrating different preference representation formats have been proposed [9–11,15,33,42,45,54,55]. In these research papers, many reasons are provided for fuzzy preference relations to be chosen as the base element of that integration. Among these reasons, it is worth noting that they are a useful tool in the aggregation of experts' preferences into group preferences [4,7,9,17,22,24,26,31,43,44,56].

GDM suffers from several disadvantages. We know that groups rarely outperform their best member. While groups have the potential to arrive at an effective decision, they often suffer from process losses. For example, groups may suffer from coordination problems. Anyone who has worked with a team of individuals on a project can attest to the difficulty of coordinating members' work or even coordinating everyone's presence in a team meeting. Furthermore, groups can suffer from groupthink. Finally, GDM takes more time compared to individual decision-making, because all members need to discuss their thoughts regarding different alternatives. It is unrealistic to expect every decision maker to be certain about the relative importance of each alternative over others. An expert may be ambiguous about the problem at hand, or he or she may lack sufficient knowledge to distinguish the extent to which some alternatives are superior to others.

The literature proposes several methods for completing incomplete fuzzy preference relations. Zai-Wu et al. [51] study a goal programming approach to complete intuitionistic fuzzy preference relations (IFPR) whose equivalent matrices are formulated to avoid the operational difficulty caused by complex operation laws in IFPR. They assume that each decision maker provides weight information to obtain priority vectors. Alonso et al. [5] give an estimation procedure for two tuple fuzzy linguistic preference relations. They give a transformation function to define additive consistency for such preference relations. Two methods for estimating missing pairwise preference values given by Fedrizzi and Giove [16] and Hererra-Viedma et al. [25] are compared by Chiclana et al. in [8]. Chiclana et al. deduced that Fedrizzi's method to estimate missing values based on the resolution of optimization is a special case of Herrera's estimation method based on known preference values. Herrera et al. proposed a method in [24] to estimate missing values in an incomplete fuzzy preference relation when $N - 1$ preference values are provided by the expert. A more general condition that includes the case where a complete row or column is given is provided in [25]. Estimated preference values that surpassed the unit interval were taken care of with a transformation function defined by Hererra-Viedma et al. in [24,25]. These transformation functions result in a complete preference relation with preference values inside the interval $[0, 1]$, but the consistency of the resultant relation is not assured. Moreover, this can void the originality of preference values given by the experts.

Herrera proposed a method in [24] to estimate missing values in an incomplete fuzzy preference relation when $(n - 1)$ preference values $\{p_{12}, p_{23}, \ldots, p_{(n-1),n}\}$ are provided by the expert. A more general condition which includes the case where a complete row or column is given is provided in [25]. Estimated preference values that surpassed the unit interval were taken care of with a transformation function defined by Hererra-Viedma et al. in [24,25]. These transformation functions result in a complete preference relation with preference values inside the interval $[0, 1]$, but the consistency of the resultant relation is not assured. Moreover, this can void the originality of preference values given by the experts.

The q-step orthopair fuzzy set was invented by Yager [50]. In this set theory, the key criterion is that the total of membership grade and non-membership grade should not be more than 1. Senapati and Yager [36] introduced and investigated the Fermatean fuzzy set (FFS) based on this concept. Senapati and Yager [37] defined and investigated the characteristics of Fermatean arithmetic means, division, and subtraction, which are novel transactions for FFS. Furthermore, Senapati and Yager [36] adapted the TOPSIS approach, which is commonly utilized in MCDM situations, to FFS. Senapati and Yager [37] extended their work by investigating numerous more operations, including subtraction, division, and Fermatean arithmetic mean operations over FFSs and using the Fermatean fuzzy weighted product model to address MCDM issues.

In [38], additional FFS aggregation operators were developed, and characteristics associated with these operators were investigated. FFSs are one of the most important concepts to accommodate more uncertainties than intuitionistic fuzzy sets (IFSs) [6] and pythagorean fuzzy sets (PFSs) [52,53] to represent fuzzy information. In this sense, FFSs contain more information than IFSs and PFSs, which have attracted the attention of many researchers [13,14,27–30,37,38]. Donghai et al. [13] proposed the notion of Fermatean fuzzy linguistic word sets in their study. These sets' operations, scoring, and accuracy functions were provided. Donghai et al. [14] proposed a new similarity metric for Fermatean fuzzy linguistic word sets. The new metric is a hybrid of the Euclidean distance measure and the cosine similarity measure. New weighted aggregated operators relevant to FFSs are defined in [38]. Fermatean hesitant fuzzy set has been defined by [29]. Shahzadi and Akram [39] proposed a novel FFSS decision support algorithm and developed new aggregated operators. Garg et al. [18] proposed new FFS type aggregated operators defined by t-norm and t-conorm. In [20,21], Hamacher-type operators for Fermatean fuzzy numbers are examined. In [29], a novel hesitant fuzzy set is known as the Fermatean hesitant fuzzy set is presented and its features are studied. The ELECTRE I approach is developed in [28] using FFS and the GDM process, in which several individuals engage at the same time. New correlation coefficients based on FFS using variance and covariance information have been provided in [30]. The usefulness of FFSs with uses such as FF soft expert knowledge, FF N-soft sets, and COVID-19 applications are demonstrated in recently published works on FFSs [1–3]. These ideas, despite all of their potential answers, have limits. Examples of these restrictions are how to specify the membership function in each specific object and shortcomings in contemplating the parametrization tool. These constraints make it difficult for decision-makers to make sound decisions during the analysis.

As stated earlier, FFSs are one of the most important concepts to accommodate more uncertainties than IFSs and PFSs to represent fuzzy information. Due to its powerfulness in describing ambiguity and indeterminacy. Let us explain the main contributions of this study: An additive consistency concept for Fermatean fuzzy preference relations (FFPRs) is presented to ensure the ranking is reasonable. A consistency index for assessing the consistency of FFPRs is proposed. When FFPRs are inconsistent, an algorithm for repairing inconsistent judgments is offered. Moreover, when FFPRs is incomplete, a model for deriving unknown judgments is given. A method for obtaining the ranking values of alternatives from an FFPR is presented. A consensus index and a consensus-reaching procedure are presented to rationalize the decision-making results because the DMs may be insufficient confidence in their judgments of FFPRs. A GDM method with incomplete FFPRS is offered to guarantee reasonable ranking orders of alternatives with high consensus levels.

1.3.1 Originality

We propose a method for estimating missing information in an expert's incomplete fuzzy preference relation using only the remaining preference values provided by that expert. By doing so, we ensure that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the expert's information. Therefore, a new decision model to deal with GDM problems with incomplete fuzzy preference relations based on the additive consistency property has been given. We have been showing that under certain conditions the incomplete fuzzy preference relation can be completed, i.e., all its missing values can be successfully estimated. It is important to point out that because no information from external sources is used, the estimated information is consistent with the original expert's opinions.

This study has some significant novelties. Considering incomplete preferences in the decision-making process provides significant advantages. With the help of this methodology, the missing information in the evaluation matrix can be completed. In the decision-making process, one of the significant problems is that decision makers sometimes may not have a clear opinion about the relationship between some factors. Because an evaluation cannot be conducted with missing information, the decision makers are forced to make evaluations for these items although they do not have sufficient information about them. This situation decreases the effectiveness of the analysis results. In this regard, owing to incomplete preferences,

missing information can be completed, which means that the decision makers do not have to evaluate the factors if they do not have opinions.

1.3.2 Work structure

This study is arranged: Section 2 presents basic information. In Section 3, the additive consistency structure for FFPRs is introduced and models are given. Section 4 is dedicated to defining a new GDM in the incomplete FR environment and examining its key features. In Section 5, an illustrative example of the given algorithm is studied.

2 Preliminaries

Now, some fundamental information that will be used in the study will be given.

Definition 2.1. For $X = \{x_1, x_2, \ldots, x_n\}$, if the set

$$S = \{(x, \zeta_S(x), \eta_S(x)) : x \in X\}$$

satisfies the following conditions, then the set S is called FFS:

$$\zeta_S, \eta_S \in [0, 1], \quad 0 \le \zeta_S^3 + \eta_S^3 \le 1.$$

$\theta_S = (1 - \zeta_S^3 + \eta_S^3)^{1/3}$ shows the hesitation degree.

The pair $(\zeta_S(x), \eta_S(x))$ in the FFS S is defined as a Fermatean fuzzy number (FFN).
Choose the FFNs $\varepsilon = (\zeta_\varepsilon, \eta_\varepsilon)$ and $\rho = (\zeta_\rho, \eta_\rho)$.

(a) $\overline{\varepsilon} = (\eta_\varepsilon, \zeta_\varepsilon)$,
(b) $\varepsilon \boxplus \rho = ((\zeta_\varepsilon^3 + \zeta_\rho^3 - \zeta_\varepsilon^3\zeta_\rho^3)^{1/3}, \eta_\varepsilon\eta_\rho)$,
(c) $\varepsilon \boxtimes \rho = (\zeta_\varepsilon\zeta_\rho, (\eta_\varepsilon^3 + \eta_\rho^3 - \eta_\varepsilon^3\eta_\rho^3)^{1/3})$,
(d) $z \cdot \varepsilon = ((1 - (1 - \zeta_\varepsilon^3)^z)^{1/3}, \eta_\varepsilon^z)$,
(e) $\varepsilon^z = (\zeta_\varepsilon^z, (1 - (1 - \eta_\varepsilon^3)^z)^{1/3})$.

Definition 2.2. [36] Consider the two FFNs $\varepsilon = (\zeta_\varepsilon, \eta_\varepsilon)$ and $\rho = (\zeta_\rho, \eta_\rho)$. For ε and ρ, the score functions $S(\varepsilon) = \zeta_\varepsilon^3 - \eta_\varepsilon^3$ and $S(\rho) = \zeta_\rho^3 - \eta_\rho^3$ and the accuracy functions $A(\varepsilon) = \zeta_\varepsilon^3 + \eta_\varepsilon^3$ and $A(\rho) = \zeta_\rho^3 + \eta_\rho^3$.

In this definition, the following situations are held:
- If $S(\varepsilon) < S(\rho)$, then $\varepsilon < \rho$,
- If $S(\varepsilon) = S(\rho)$, $A(\varepsilon) < A(\rho)$, then $\varepsilon < \rho$,
- If $S(\varepsilon) = S(\rho)$, $A(\varepsilon) = A(\rho)$, then $\varepsilon = \rho$.

Examining the Definition 2.2 will reveal that these conditions will not be met for every *FFN*. To solve this problem, the definition of the new score function can be given as follows:

Definition 2.3. [12] For a FFN $\varepsilon = (\zeta_\varepsilon, \eta_\varepsilon)$, if

$$\mathscr{S}(\varepsilon) = \frac{[(1 - \eta_\varepsilon)^3 + 1 - \eta_\varepsilon^3]^{1/3}}{[(1 - \eta_\varepsilon)^3 + 1 - \eta_\varepsilon^3]^{1/3} + [(1 - \zeta_\varepsilon)^3 + 1 - \zeta_\varepsilon^3]^{1/3}},$$

where $0 \le \mathscr{S}(\varepsilon) \le 1$, then $\mathscr{S}$ is called score function.

Let $\varepsilon = (\zeta_\varepsilon, \eta_\varepsilon)$ be a $\mathfrak{FFN}$
If $\zeta_\varepsilon = \eta_\varepsilon$, then $\mathscr{S}(\varepsilon) = 1/2$;
If $\zeta_\varepsilon > \eta_\varepsilon$, then $\mathscr{S}(\varepsilon) > 1/2$;
If $\zeta_\varepsilon < \eta_\varepsilon$, then $\mathscr{S}(\varepsilon) < 1/2$.
Take two FFNs $\varepsilon = (\zeta_\varepsilon, \eta_\varepsilon)$ and $\rho = (\zeta_\rho, \eta_\rho)$.

(i) If $\zeta_\varepsilon = \zeta_\rho$, $\eta_\varepsilon < \eta_\rho$, then $\mathscr{S}(\varepsilon) > \mathscr{S}(\rho)$;
If $\eta_\varepsilon = \eta_\rho$, $\zeta_\varepsilon > \zeta_\rho$, then $\mathscr{S}(\varepsilon) > \mathscr{S}(\rho)$.

(ii) If $\zeta_\varepsilon > \zeta_\rho$, $\eta_\varepsilon < \eta_\rho$, then $\mathscr{S}(\varepsilon) > \mathscr{S}(\rho)$.

Example 2.4. Let $F = (0.77, 0.56)$ and $G = (0.6, 0.7)$ be two FFSs. By using the Definition 2.3, we have $\mathscr{S}(F) = 0.54$ and $\mathscr{S}(G) = 0.48$. For $F = (0.77, 0.56)$, since $\zeta_F > \eta_F$, then $\mathscr{S}(F) = 0.54 > 1/2$. For $G = (0.6, 0.7)$, since $\zeta_G < \eta_G$, then $\mathscr{S}(G) = 0.48 < 1/2$. Furthermore, since $\zeta_F > \zeta_G$ and $\eta_F < \eta_G$, then $\mathscr{S}(F) = 0.54 > \mathscr{S}(G) = 0.48$.

Definition 2.5. [12] Let $\varepsilon = (\zeta_\varepsilon, \eta_\varepsilon)$ and $\rho = (\zeta_\rho, \eta_\rho)$ be two FFNs and $\mathscr{S}(\rho)$, $\mathscr{S}(\varepsilon)$ be two score functions. Considering $A(\rho)$ and $A(\varepsilon)$ as accuracy functions, the following cases hold:

(i) $\mathscr{S}(\varepsilon) < \mathscr{S}(\rho) \Rightarrow \varepsilon < \rho$;
(ii) $\mathscr{S}(\varepsilon) = \mathscr{S}(\rho) \Rightarrow$
  (a) $A(\varepsilon) < A(\rho) \Rightarrow \varepsilon < \rho$
  (b) $A(\varepsilon) = A(\rho) \Rightarrow \varepsilon = \rho$.

Example 2.6. Let $F = (0.77, 0.56)$ and $G = (0.78, 0.55)$ be two FFSs. By using the Definition 2.3, we have $\mathscr{S}(F) = 0.54$ and $\mathscr{S}(G) = 0.54$. Since $\mathscr{S}(F) = \mathscr{S}(G)$, let's use the accuracy function. Then, $A(F) = 0.636$ and $A(G) = 0.641$. That is, $A(F) < A(G) \Rightarrow F < G$.

Definition 2.7. [12] A matrix $M = (m_{ij})_{n\times n}$ on an alternatives X is a FFPR ($\mathfrak{FR}$) in which each element $m_{ij} = (\zeta_{ij}, \eta_{ij})$ is characterized by ζ_{ij}, η_{ij} from $X \times X$ to unit interval and hesitancy function θ_{ij} from $X \times X$ to unit interval and $\theta_{ij} = (1 - (\zeta_{ij})^3 - (\eta_{ij})^3)^{1/3}$ with the conditions (for $i, j = 1, 2, \ldots, n$):

$$\zeta_{ij} = \eta_{ji}, \quad \zeta_{ji} = n_{ij}, \quad \zeta_{ii} = \eta_{ii} = 2^{-1/3}, \quad 0 \le (\zeta_{ij})^3 + (\eta_{ij})^3 \le 1,$$
$$0 \le \zeta_{ij}, \eta_{ij}, \theta_{ij} \le 1, \quad \text{for all}.$$

Definition 2.8. [12] A matrix $M = (m_{ij})_{n\times n}$ on an alternatives X is a $\mathfrak{FR}$ in which each element $m_{ij} = (\zeta_{ij}, \eta_{ij})$ $(1 \le i \le n; 1 \le j \le n)$ is defined by ζ_{ij}, η_{ij} from $X \times X$ to unit interval. If the following condition is equivalent, then a $\mathfrak{FR}$ $M = (m_{ij})_{n\times n}$ is called additive consistent ($\mathfrak{ACt}$): For $i, j, k = 1, 2, \ldots, n$,

$$\zeta_{ij}^3 + \zeta_{jk}^3 + \zeta_{ik}^3 = \eta_{ij}^3 + \eta_{jk}^3 + \eta_{ik}^3.$$

The definition of $\mathfrak{AC}$ of $\mathfrak{FR}$ can be given as follows:

For $\mathfrak{FR}$ $M = (m_{ij})_{n\times n}$, if

$$\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 = \eta_{ij}^3 + \eta_{jk}^3 + \zeta_{ik}^3,$$

then the $\mathfrak{FR}M = (m_{ij})_{n\times n}$ is called $\mathfrak{ACt}$.

Based on this information, the following definition is given:

Definition 2.9. [12] Let $M = (m_{ij})_{n\times n}$ be a $\mathfrak{FR}$, where $m_{ij} = (\zeta_{ij}, \eta_{ij})$ $1 \le i \le n$, $1 \le j \le n$. An additive consistency index $\mathfrak{ACI}$ of the $\mathfrak{FR}$ is defined as follows:

$$\mathfrak{ACI}(M) = 1 - (2/n(n-1)(n-2)) \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} |\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3|,$$

where $\mathfrak{ACI}(M) \in [0, 1]$.

Based on the Definition 2.9, $\mathfrak{FR}$ M is a $\mathfrak{FR}$ if and only if $\mathfrak{ACI}(M) = 1$.

Example 2.10. Let $M = (m_{ij})_{3\times 3}$ be given as follows:

$$M = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.4, 0.9) & (0.2, 0.8) \\ (0.9, 0.4) & (2^{-1/3}, 2^{-1/3}) & (0.6, 0.7) \\ (0.8, 0.2) & (0.8, 0.2) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

Then $\mathfrak{ACI}(M) = 1 - \frac{2}{3.2.1}(0.4 + 0.6 - 0.9 - 0.7 - 0.2 + 0.8) = 1 - 0 = 1$.

Following the proposed additive consistency index, we offer the following definition:

Definition 2.11. A $\mathfrak{FR}$ M is acceptable additive consistent if $\mathfrak{ACI}(M) \geq \alpha$, where α is the given threshold and $\alpha \in [0, 1]$.

Example 2.12. Let $\alpha = 0.1$ and $M = (m_{ij})_{3\times 3}$ be given as follows:

$$M = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.8, 0.3) & (0.4, 0.6) \\ (0.3, 0.8) & (2^{-1/3}, 2^{-1/3}) & (0.6, 0.5) \\ (0.6, 0.4) & (0.5, 0.6) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

Then $\mathfrak{ACI}(M) = 1 - \frac{2}{3.2.1}(0.8 + 0.6 - 0.3 - 0.5 - 0.4 + 0.6) = 1 - 0.8 = 0.2$. Since $\mathfrak{ACI}(M) > \alpha$, $\mathfrak{FR}M = (m_{ij})_{3\times 3}$ is acceptable additive consistency.

3 Additive consistency for $\mathfrak{FR}$

In this section, we define additive consistency for $\mathfrak{FR}$s and discuss several appealing properties. Then, we propose some models for dealing with $\mathfrak{FR}$s that are incomplete or inconsistent.

For a $\mathfrak{FR}$ $M = (m_{ij})_{n\times n}$, where $1 \leq i \leq n$, $1 \leq j \leq n$, if some judgments in the $\mathfrak{FR}$ M is missing, then M is called an $\mathfrak{IFR}$. If there are some values in $[0, 1]$ for unknown judgments, which let the $\mathfrak{IFR}$ M be $\mathfrak{ACt}$, then we can derive equation (1). However, we cannot guarantee that equation (1) holds, i.e.,

$$\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 \neq \eta_{ij}^3 + \eta_{jk}^3 + \zeta_{ik}^3,$$

for some triples (i, j, k) with $i < j < k$. Thus, we relax equation (1) by introducing positive deviation variables d_{ijk}^+ and d_{ijk}^-, where

$$d_{ijk}^+ = \frac{|\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3| + (\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3)}{2}$$
$$d_{ijk}^- = \frac{|\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3| - (\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3)}{2}$$

and

$$\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3 - d_{ijk}^+ + d_{ijk}^- = 0,$$

for each triple (i, j, k) with $i < j < k$, where $d_{ij}^+, d_{ij}^- \geq 0$. Furthermore, we build the following model to determine unknown values in the $\mathfrak{FR}$ $M = (m_{ij})_{n\times n}$, where $m_{ij} = (\zeta_{ij}, \eta_{ij})$ $1 \leq i \leq n$, $1 \leq j \leq n$, based on equation (1):

$$h = \min \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} (d_{ijk}^{+} + d_{ijk}^{-})$$

$$\text{s.t.} \begin{cases} \zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3 - d_{ijk}^{+} + d_{ijk}^{-} = 0, \ i, j, k = 1, 2, \ldots, m, \quad i < j < k, \\ d_{ijk}^{+}, d_{ijk}^{-} \geq 0, \ i, j, k = 1, 2, \ldots, m, \quad i < j < k, \\ \zeta_{ij} \in [0, 1], \zeta_{ij}^3 + \eta_{ij}^3 \leq 1, \quad \text{if } (i, j) \in Z_\zeta \quad \text{and} \quad (i, j) \notin Z_\eta, \\ \eta_{ij} \in [0, 1], \zeta_{ij}^3 + \eta_{ij}^3 \leq 1, \quad \text{if } (i, j) \notin Z_\zeta \quad \text{and } (i, j) \in Z_\eta, \\ \zeta_{ij}, \eta_{ij} \in [0, 1], \zeta_{ij}^3 + \eta_{ij}^3 \leq 1, \quad \text{if } (i, j) \in Z_\zeta \quad \text{and} \quad (i, j) \in Z_\eta, \end{cases} \tag{1}$$

where $Z_\zeta = \{(i, j) : \zeta_{ij} \text{ is missing}, \ (i, j = 1, 2, \ldots, n; i < j)\}$ and $Z_\eta = \{(i, j) : \eta_{ij} \text{ is missing}, \ (i, j = 1, 2, \ldots, n; i < j)\}$. Solving the model 1 yields the optimal solutions, shown by $\zeta_{ij}^*((i, j) \in Z_\zeta)$, $\eta_{ij}^*((i, j) \in Z_\eta)$. Based on these optimal solutions, a complete $\mathfrak{FR}$ $M = (m_{ij})_{n\times n}$ is constructed, where

$$m_{ij} = (\zeta_{ij}, \eta_{ij}) = \begin{cases} (\zeta_{ij}, \eta_{ij}), & \text{if } (i, j) \notin Z_\zeta \quad \text{and} \quad (i, j) \notin Z_\eta, \\ (\zeta_{ij}^*, \eta_{ij}), & \text{if } (i, j) \in Z_\zeta \quad \text{and} \quad (i, j) \notin Z_\eta, \\ (\zeta_{ij}, \eta_{ij}^*), & \text{if } (i, j) \notin Z_\zeta \quad \text{and} \quad (i, j) \in Z_\eta, \\ (\zeta_{ij}^*, \eta_{ij}^*), & \text{if } (i, j) \in Z_\zeta \quad \text{and} \quad (i, j) \in Z_\eta, \\ (2^{-1/3}, 2^{-1/3}), & \text{if } i = j, \\ (\eta_{ji}, \zeta_{ji}), & \text{if } i > j, \end{cases} \tag{2}$$

for $1 \leq i \leq n; 1 \leq j \leq n$.

Let $M = (m_{ij})_{n\times n}$ be a complete $\mathfrak{FR}$, $m_{ij} = (\zeta_{ij}, \eta_{ij})$, $1 \leq i \leq n$; $1 \leq j \leq n$. When M is acceptable and consistent, then we obtain

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{j=k+1}^{n} |\zeta_{ij}^3 + \zeta_{jk}^3 + \eta_{ik}^3 - \eta_{ij}^3 - \eta_{jk}^3 - \zeta_{ik}^3| \leq \frac{n(n-1)(n-2)(1-\alpha)}{2}. \tag{3}$$

$\mathfrak{FR}$s presented by DMs are generally unacceptable and inconsistent, namely, equation (3) does not hold. To produce the ranking orders of alternatives reasonably, we must adjust original judgments of $\mathfrak{FR}$s. We propose a model to acquire an acceptable consistent $\mathfrak{FR}$ $M' = (m_{ij}')_{n\times n}$, where $m_{ij}' = (\zeta_{ij}, \eta_{ij})$ $i, j = 1, 2, \ldots, n$ and $i < j$, shown as follows:

$$f = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (|\zeta_{ij} - \zeta_{ij}'| + |\eta_{ij} - \eta_{ij}'|)$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{j=k+1}^{n} |(\zeta_{ij}')^3 + (\zeta_{jk}')^3 + (\eta_{ik}')^3 - (\eta_{ij}')^3 - (\eta_{jk}')^3 - (\zeta_{ik}')^3| \leq \frac{n(n-1)(n-2)(1-\alpha)}{2}, \\ 0 \leq \zeta_{ij}' \leq 1, \ ij, =1, 2, \ldots, n, \ i < j, \\ 0 \leq \eta_{ij}' \leq 1, \ ij, =1, 2, \ldots, n, \ i < j, \\ 0 \leq (\zeta_{ij}')^3 + (\eta_{ij}')^3 \leq 1, \ ij, =1, 2, \ldots, n, \ i < j. \end{cases} \tag{4}$$

Let $s_{ij}^{+} = \frac{|\zeta_{ij} - \zeta_{ij}'| + (\zeta_{ij} - \zeta_{ij}')}{2}$ and $s_{ij}^{-} = \frac{|\zeta_{ij} - \zeta_{ij}'| - (\zeta_{ij} - \zeta_{ij}')}{2}$. Then, we have $|\zeta_{ij} - \zeta_{ij}'| = s_{ij}^{+} + s_{ij}^{-}$, $\zeta_{ij} - \zeta_{ij}' = s_{ij}^{+} - s_{ij}^{-}$. Let $t_{ij}^{+} = \frac{|\eta_{ij} - \eta_{ij}'| + (\eta_{ij} - \eta_{ij}')}{2}$ and $t_{ij}^{-} = \frac{|\eta_{ij} - \eta_{ij}'| - (\eta_{ij} - \eta_{ij}')}{2}$. Then, we obtain $|\eta_{ij} - \eta_{ij}'| = t_{ij}^{+} + t_{ij}^{-}$ and $\eta_{ij} - \eta_{ij}' = t_{ij}^{+} - t_{ij}^{-}$.

Let $u_{ijk}^{+} = \frac{|(\zeta_{ij}')^3 + (\zeta_{jk}')^3 + (\eta_{ik}')^3 - (\eta_{ij}')^3 - (\eta_{jk}')^3 - (\zeta_{ik}')^3| + ((\zeta_{ij}')^3 + (\zeta_{jk}')^3 + (\eta_{ik}')^3 - (\eta_{ij}')^3 - (\eta_{jk}')^3 - (\zeta_{ik}')^3)}{2}$ and

let $u_{ijk}^{-} = \dfrac{|(\zeta_{ij}')^3 + (\zeta_{jk}')^3 + (\eta_{ik}')^3 - (\eta_{ij}')^3 - (\eta_{jk}')^3 - (\zeta_{ik}')^3| - ((\zeta_{ij}')^3 + (\zeta_{jk}')^3 + (\eta_{ik}')^3 - (\eta_{ij}')^3 - (\eta_{jk}')^3 - (\zeta_{ik}')^3)}{2}$.

Then, we obtain $|(\zeta_{ij}')^3 + (\zeta_{jk}')^3 + (\eta_{ik}')^3 - (\eta_{ij}')^3 - (\eta_{jk}')^3 - (\zeta_{ik}')^3| = u_{ijk}^{+} + u_{ijk}^{-}$ and $(\zeta_{ij}')^3 + (\zeta_{jk}')^3 + (\eta_{ik}')^3 - (\eta_{ij}')^3 - (\eta_{jk}')^3 - (\zeta_{ik}')^3 = u_{ijk}^{+} - u_{ijk}^{-}$.

Therefore, the model 4 is transformed as follows:

$$f = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (s_{ij}^{+} + s_{ij}^{-} + t_{ij}^{+} + t_{ij}^{-})$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{j=k+1}^{n} (u_{ijk}^{+} + u_{ijk}^{-})(n(n-1)(n-2)(1-\alpha)/2), \\ (\zeta_{ij} + s_{ij}^{-} - s_{ij}^{+})^3 + (\zeta_{jk} + s_{jk}^{-} - s_{jk}^{+})^3 + (\eta_{ik} + t_{ik}^{-} - t_{ik}^{+})^3 - (\eta_{ij} + t_{ij}^{-} - t_{ij}^{+})^3 - (\eta_{jk} + t_{jk}^{-} - t_{jk}^{+})^3 \\ - (\zeta_{ik} + s_{ik}^{-} - s_{ik}^{+})^3 + u_{ijk}^{-} - u_{ijk}^{+} = 0, \ i, j, k = 1, 2, \ldots, n, \ i < j < k, \\ s_{ij}^{-}, s_{ij}^{+}, t_{ij}^{-}, t_{ij}^{+} \geq 0, \ i, j = 1, 2, \ldots, n, \ i < j, \\ u_{ijk}^{-}, u_{ijk}^{+} \geq 0, \ i, j, k = 1, 2, \ldots, n, \ i < j < k, \\ 0 \leq \zeta_{ij} + s_{ij}^{-} - s_{ij}^{+} \leq 1, \quad i, j = 1, 2, \ldots, n, \ i < j, \\ 0 \leq \eta_{ij} + t_{ij}^{-} - t_{ij}^{+} \leq 1, \quad i, j = 1, 2, \ldots, n, \ i < j, \\ 0 \leq (\zeta_{ij} + s_{ij}^{-} - s_{ij}^{+})^3 + (\eta_{ij} + t_{ij}^{-} - t_{ij}^{+})^3 \leq 1, \ i, j = 1, 2, \ldots, n, \ i < j. \end{cases} \tag{5}$$

After solving the model 5, we can derive the optimal solutions, shown by $s_{ij}^{*-}, s_{ij}^{*+}, t_{ij}^{*-}, t_{ij}^{*+}, u_{ijk}^{*-}, u_{ijk}^{*+}$ $(i, j, k = 1, 2, \ldots, n \quad i < j < k)$. Then, we can gain a modified acceptable additive consistent $\mathfrak{FR}$ $M' = (m_{ij}')n \times n$, where

$$m_{ij}' = (\zeta_{ij}', \eta_{ij}') = \begin{cases} (\zeta_{ij} + s_{ij}^{*-} - s_{ij}^{*+}, \eta_{ij} + t_{ij}^{*-} - t_{ij}^{*+}), & \text{if } i < j, \\ (2^{-1/3}, 2^{-1/3}), & \text{if } i = j, \\ (\eta_{ji} + t_{ji}^{*-} - t_{ji}^{*+}, \zeta_{ji} + s_{ji}^{*-} - s_{ji}^{*+}), & \text{if } i > j. \end{cases} \tag{6}$$

On the premise of keeping the minimum distance between the initial $\mathfrak{FR}$ and the modified $\mathfrak{FR}$, we hope that the number of adjusted elements in the adjusted $\mathfrak{FR}$ is as small as possible. Therefore, we also propose a model to modify an unacceptable additive consistent $\mathfrak{FR}$ shown as follows:

$$e = \max \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (v_{ij} + y_{ij}),$$

$$\text{s.t.} \begin{cases} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} (u_{ijk}^{+} + u_{ijk}^{-})(n(n-1)(n-2)(1-\alpha)/2), \\ (\zeta_{ij} + s_{ij}^{-} - s_{ij}^{+})^3 + (\zeta_{jk} + s_{jk}^{-} - s_{jk}^{+})^3 + (\eta_{ik} + t_{ik}^{-} - t_{ik}^{+})^3 - (\eta_{ij} + t_{ij}^{-} - t_{ij}^{+})^3 - (\eta_{jk} + t_{jk}^{-} - t_{jk}^{+})^3 \\ - (\zeta_{ik} + s_{ik}^{-} - s_{ik}^{+})^3 + u_{ijk}^{-} - u_{ijk}^{+} = 0, \quad i, j, k = 1, 2, \ldots, n, \quad i < j < k, \\ \zeta_{ij} + s_{ij}^{-} + s_{ij}^{+} = (1 - v_{ij})(\zeta_{ij} + s_{ij}^{-} + s_{ij}^{+}) + v_{ij}\zeta_{ij}, \ i, j = 1, 2, \ldots, n, \quad i < j, \\ \eta_{ij} + t_{ij}^{-} + t_{ij}^{+} = (1 - y_{ij})(\eta_{ij} + t_{ij}^{-} + t_{ij}^{+}) + y_{ij}\zeta_{ij}, \ i, j = 1, 2, \ldots, n \quad i < j, \\ v_{ij}, y_{ij} \text{ in } \{0, 1\}, \quad i, j = 1, 2, \ldots, n \quad i < j \\ f^* = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (s_{ij}^{-} + s_{ij}^{+} + t_{ij}^{-} + t_{ij}^{+}), \\ s_{ij}^{-}, s_{ij}^{+}, t_{ij}^{-}, t_{ij}^{+} \geq 0, \ i, j = 1, 2, \ldots, n \quad i < j, \\ u_{ijk}^{-}, u_{ijk}^{+} \geq 0, \ i, j, k = 1, 2, \ldots, n \quad i < j < k, \\ 0 \leq \zeta_{ij} + s_{ij}^{-} - s_{ij}^{+} \leq 1, \ i, j = 1, 2, \ldots, n \quad i < j, \\ 0 \leq \eta_{ij} + t_{ij}^{-} - t_{ij}^{+} \leq 1, \quad i, j = 1, 2, \ldots, n \quad i < j, \\ 0 \leq (\zeta_{ij} + s_{ij}^{-} - s_{ij}^{+})^3 + (\eta_{ij} + t_{ij}^{-} - t_{ij}^{+})^3 \leq 1, \ i, j, k = 1, 2, \ldots, n \quad i < j < k, \end{cases} \tag{7}$$

where $s_{ij}^{-}, s_{ij}^{+}, t_{ij}^{-}, t_{ij}^{+}$, u_{ijk}^{-}, u_{ijk}^{+} $(i, j, k = 1, 2, \ldots, n\ i < j < k)$ are given as model 6. $f^* = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}$ $(s_{ij}^{*-} + s_{ij}^{*+} + t_{ij}^{*-} + t_{ij}^{*+})$, and $s_{ij}^{*-}, s_{ij}^{*+}, t_{ij}^{*-}, t_{ij}^{*+}$ are the optimal values obtained by the Model 6.

In the model 7, v_{ij} is a $0-1$ variable, which is used to indicate whether the MD ζ_{ij} in the original $\mathfrak{FR}$ M is adjusted, where $m_{ij} = (\zeta_{ij}, \eta_{ij})$ $1 \le i \le n,\ 1 \le j \le n$. If ζ_{ij} is adjusted, then $v_{ij} = 0$. Otherwise, $v_{ij} = 1$. Similarly, y_{ij} is a $0-1$ variable, which is used to indicate whether the ND η_{ij} in the original $\mathfrak{FR}$ M is adjusted, where $m_{ij} = (\zeta_{ij}, \eta_{ij})$ $1 \le i \le n,\ 1 \le j \le n$. If η_{ij} is adjusted, then $y_{ij} = 0$. Otherwise, $y_{ij} = 1$. After solving the model 7, we can obtain the optimal solutions, denoted by $s_{ij}^{\diamond-}, s_{ij}^{\diamond+}, t_{ij}^{\diamond-}, t_{ij}^{\diamond+}$ $i, j = 1, 2, \ldots, n,\ i < j,\ u_{ijk}^{\diamond-}, u_{ijk}^{\diamond+}$ $i, j, k = 1, 2, \ldots, n,\ i < j < k$, and $y_{ij}^{\diamond}, y_{ij}^{\diamond}$ $i, j = 1, 2, \ldots, n,\ i < j$. Then, we can also acquire a revised acceptable additive consistent $\mathfrak{FR}$ $M' = (m_{ij}')_{n\times n}$, where

$$m_{ij}' = (\zeta_{ij}', \eta_{ij}') = \begin{cases} (\zeta_{ij} + s_{ij}^{\diamond-} - s_{ij}^{\diamond+}, \eta_{ij} + t_{ij}^{\diamond-} - t_{ij}^{\diamond+}), & \text{if } i < j, \\ (2^{-1/3}, 2^{-1/3}), & \text{if } i = j, \\ (\eta_{ji} + t_{ji}^{\diamond-} - t_{ji}^{\diamond+}, \zeta_{ji} + s_{ji}^{\diamond-} - s_{ji}^{\diamond+}), & \text{if } i > j, \end{cases} \tag{8}$$

where $1 \le i \le n, 1 \le j \le n$.

Definition 3.1. Let ω be the FF priority weight vector of a $\mathfrak{FR}$ $M = (m_{ij})_{n\times n}$, where $m_{ij} = (\zeta_{ij}, \eta_{ij})$, $1 \le i \le n,\ 1 \le j \le n$, $\omega_i = (\zeta_{\omega_i}, \eta_{\omega_i})$ is a FFN with the conditions $\zeta_{\omega_i}, \eta_{\omega_i} \in [0, 1]$ and $\zeta_{\omega_i} + \eta_{\omega_i} \le 1$. The FF priority weight vector is called a normalized if it satisfies $\sum_{j=1}^{n}\zeta_{\omega_j}^3 \le \eta_{\omega_i}^3$ and $\zeta_{\omega_i}^3 + n - 2 \ge \sum_{j=1;j\neq i}^{n}\eta_{\omega_j}^3$ for all $i = 1, 2, \ldots, n$.

According to FF priority weight vector ω, we construct a matrix $P = (p_{ij})_{n\times n}$, where

$$p_{ij} = \left(\zeta_{p_i j}, \eta_{p_i j}\right) = \begin{cases} (2^{-1/3}, 2^{-1/3}), & \text{if } i = j, \\ ((2^{-1}\zeta_{\omega_i}^3 + 2^{-1}\eta_{\omega_i}^3)^{1/3}, (2^{-1}\eta_{\omega_i}^3 + 2^{-1}\zeta_{\omega_i}^3)^{1/3}), & \text{if } i \neq j. \end{cases} \tag{9}$$

Definition 3.2. The matrix $P = (p_{ij})_{n\times n}$, where p_{ij} is defined in equation (10), is an additive consistent $\mathfrak{FR}$.

Proof. (i) By equation (10), we have,

$$\begin{cases} 0 \le \zeta_{m_{ij}} = (2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_i}^3)^{1/3} \le 1, \\ 0 \le \eta_{m_{ij}} = (2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_i}^3)^{1/3} \le 1, \\ \zeta_{\omega_i}^3 = \eta_{\omega_i}^3 = (2^{-1/3})^{1/3}, \\ \zeta_{\omega_i}^3 = (2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_i}^3)^{1/3} = (2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_i}^3)^{1/3} = \eta_{\omega_i}^3, \\ \eta_{\omega_i}^3 = (2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_i}^3)^{1/3} = (2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_i}^3)^{1/3} = \zeta_{\omega_i}^3, \\ \zeta_{\omega_i}^3 + \eta_{\omega_i}^3 = ((2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_j}^3)^{1/3})^3 + ((2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_j}^3)^{1/3})^3 \\ = 2^{-1}\zeta_{\omega_i}^3 + 2^{-1}\eta_{\omega_j}^3 + 2^{-1}\eta_{\omega_i}^3 + 2^{-1}\zeta_{\omega_j}^3 = 2^{-1}(\zeta_{\omega_i}^3 + \eta_{\omega_i}^3) + 2^{-1}(\zeta_{\omega_j}^3 + \eta_{\omega_j}^3) \le 1. \end{cases} \tag{10}$$

Following Definition 2.7, we can see that P is a $\mathfrak{FR}$.

(ii) For all $i, j, k = 1, 2, \ldots, n$ with $i < j < k$, we derive,

$$\begin{aligned} \zeta_{p_i j}^3 + \zeta_{p_j k}^3 + \zeta_{p_i k}^3 &= ((2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_j}^3)^{1/3})^3 + ((2^{-1/3}\zeta_{\omega_j}^3 + 2^{-1/3}\eta_{\omega_k}^3)^{1/3})^3 + ((2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_k}^3)^{1/3})^3 \\ &= 2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_j}^3 + 2^{-1/3}\zeta_{\omega_j}^3 + 2^{-1/3}\zeta_{\omega_k}^3 + 2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_k}^3 \\ &= ((2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_j}^3)^{1/3})^3 + ((2^{-1/3}\eta_{\omega_j}^3 + 2^{-1/3}\zeta_{\omega_k}^3)^{1/3})^3 + ((2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_k}^3)^{1/3})^3 \\ &= \eta_{p_i j}^3 + \eta_{p_j k}^3 + \eta_{p_i k}^3. \end{aligned}$$

Following Definition 2.8, P is additive consistent.

From (i) and (ii), P is additive consistent. □

Theorem 3.3. *Assume that $M = (m_{ij})_{n\times n}$ is a $\mathfrak{FR}$, where $m_{ij} = (\zeta_{ij}, \eta_{ij}), 1 \le i \le n,\ 1 \le j \le n$, then MI is additive consistent, if there exists a normalized FF priority weight vector ω ($\omega_i = (\zeta_{\omega_i}, \eta_{\omega_i})$) such that*

$$(\zeta_{ij}, \eta_{ij}) = \begin{cases} (2^{-1/3}, 2^{-1/3}), & \text{if } i = j, \\ ((2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_j}^3)^{1/3}, (2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_j}^3)^{1/3}), & \text{if } i \neq j. \end{cases} \tag{11}$$

Based on equation (11), if a $\mathfrak{FR}$ $M = (m_{ij})_{n\times n}$ is additive consistent, where $m_{ij} = (\zeta_{ij}, \eta_{ij})$, $1 \le i \le n$, $1 \le j \le n$, we have:

$$\begin{cases} \zeta_{ij} = (2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_j}^3)^{1/3}, \\ \eta_{ij} = (2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_j}^3)^{1/3}, \end{cases} \tag{12}$$

for all $i, j = 1, 2, \ldots, n$ with $i < j$. Following equation (12), the smaller the value of $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\zeta_{ij} - (2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_i}^3)^{1/3}| + |\eta_{ij} - (2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_i}^3)^{1/3}|)$, the higher the additive consistency level will be. Hence, we offer the following model to derive a priority weight vector ω:

$$k = \min \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\zeta_{ij} - (2^{-1/3}\zeta_{\omega_i}^3 + 2^{-1/3}\eta_{\omega_i}^3)^{1/3}| + |\eta_{ij} - (2^{-1/3}\eta_{\omega_i}^3 + 2^{-1/3}\zeta_{\omega_i}^3)^{1/3}|)$$

$$\text{s.t.} \begin{cases} \zeta_{\omega_i}, \eta_{\omega_i} \in [0, 1], \quad i = 1, 2, \ldots, n, \\ \zeta_{\omega_i}^3 + \eta_{\omega_i}^3 \le 1, \quad i = 1, 2, \ldots, n, \\ \sum_{j=1; i\neq j}^{n} \zeta_{\omega_j}^3 \le \eta_{\omega_i}^3, \quad i = 1, 2, \ldots, n, \\ \zeta_{\omega_i}^3 + n - 2 \ge \sum_{j=1; i\neq j}^{n} \eta_{\omega_j}^3, \quad i = 1, 2, \ldots, n, . \end{cases} \tag{13}$$

Example 3.4. Let $A = \{a_1, a_2, a_3, a_4\}$ be set of alternatives, and let $M = (m_{ij})_{4\times 4}$ be an incomplete $\mathfrak{FR}$ on A. Let M be given as follows:

$$M = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.28, 0.73) & (-,0.61) & (0.63, 0.53) \\ (0.73, 0.28) & (2^{-1/3}, 2^{-1/3}) & (0.54, 0.88) & (-,-) \\ (0.61, -) & (0.88, 0.54) & (2^{-1/3}, 2^{-1/3}) & (0.48, 0.87) \\ (0.53, 0.63) & (-,-) & (0.87, 0.48) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

By substituting $M = (m_{ij})_{4\times 4}$ into the model 1, we may obtain the optimal solutions $\zeta_{13}^*, \zeta_{24}^*, \eta_{24}^*$, where $\zeta_{13}^* = 0.8453$, $\zeta_{24}^* = 0.7185$, and $\eta_{24}^* = 0.7736$. After substituting the obtained three optimal solutions $\zeta_{13}^* = 0.8453$, $\zeta_{24}^* = 0.7185$ and $\eta_{24}^* = 0.7736$ into equation (2), we obtain a complete $\mathfrak{FR}$ $M = (m_{ij})_{4\times 4}$ shown as follows:

$$M = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.28, 0.73) & (0.8453, 0.61) & (0.63, 0.53) \\ (0.73, 0.28) & (2^{-1/3}, 2^{-1/3}) & (0.54, 0.88) & (0.7185, 0.7736) \\ (0.61, 0.8453) & (0.88, 0.54) & (2^{-1/3}, 2^{-1/3}) & (0.48, 0.87) \\ (0.53, 0.63) & (0.7736, 0.7185) & (0.87, 0.48) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

By using equation (1), we obtain $\mathfrak{ACI}(M) = 0.7934$. Let $\alpha = 0.8$. Because $\mathfrak{ACI}(M) < \alpha$, M is unacceptable additive consistent. Plug M into the model 5 and solve this equation, and we can derive its optimal objective

function value $f^* = 0.2975$. Hence, plug M and f^* value into the model 7 and solve this model, and we can derive its optimal solutions as follows:

$$
\begin{aligned}
&s_{12}^{\diamond-} = 0, \quad s_{13}^{\diamond-} = 0, \quad s_{14}^{\diamond-} = 0 \quad s_{23}^{\diamond-} = 0.0176 \quad s_{24}^{\diamond-} = 0 \quad s_{34}^{\diamond-} = 0\\
&s_{12}^{\diamond+} = 0, \quad s_{13}^{\diamond+} = 0, \quad s_{14}^{\diamond+} = 0.0106 \quad s_{23}^{\diamond+} = 0, \quad s_{24}^{\diamond+} = 0, \quad s_{34}^{\diamond+} = 0,\\
&t_{12}^{\diamond-} = 0, \quad t_{13}^{\diamond-} = 0, \quad t_{14}^{\diamond-} = 0, \quad t_{23}^{\diamond-} = 0, \quad t_{24}^{\diamond-} = 0, \quad t_{34}^{\diamond-} = 0,\\
&t_{12}^{\diamond+} = 0.0953, \quad t_{13}^{\diamond+} = 0, \quad t_{14}^{\diamond+} = 0, \quad t_{23}^{\diamond+} = 0.0248, \quad t_{24}^{\diamond+} = 0.1284 \quad t_{34}^{\diamond+} = 0.0641\\
&u_{123}^{\diamond-} = 0.58, \quad u_{124}^{\diamond-} = 0.576, \quad u_{134}^{\diamond-} = 0, \quad u_{234}^{\diamond-} = 0, u_{123}^{\diamond+} = 0, \quad u_{124}^{\diamond+} = 0, \quad u_{134}^{\diamond+} = 0, \quad u_{234}^{\diamond+} = 0,\\
&v_{12}^{\diamond} = 1, \quad v_{13}^{\diamond} = 1, \quad v_{14}^{\diamond} = 0, \quad v_{23}^{\diamond} = 0, \quad v_{24}^{\diamond} = 1, \quad v_{34}^{\diamond} = 1,\\
&y_{12}^{\diamond} = 0, \quad y_{13}^{\diamond} = 1, \quad y_{14}^{\diamond} = 1, \quad y_{23}^{\diamond} = 0, \quad y_{24}^{\diamond} = 0, \quad y_{34}^{\diamond} = 0.
\end{aligned}
$$

Furthermore, after inserting the aforementioned optimal solutions into equation (8), we can obtain the adjusted $\mathfrak{FR}$ M with the acceptable additive consistency, where

$$
M = \begin{bmatrix}
(2^{-1/3}, 2^{-1/3}) & (0.28, 0.6347) & (0.8453, 0.61) & (0.62, 0.53)\\
(0.6347, 0.28) & (2^{-1/3}, 2^{-1/3}) & (0.5576, 0.8552) & (0.7185, 0.6452)\\
(0.8453, 0.61) & (0.8552, 0.5576) & (2^{-1/3}, 2^{-1/3}) & (0.48, 0.806)\\
(0.53, 0.63) & (0.6452, 0.7185) & (0.806, 0.48) & (2^{-1/3}, 2^{-1/3})
\end{bmatrix}.
$$

After inserting the adjusted $\mathfrak{FR}$ M into the model 13 and solving it, we can obtain the priority vector ω, shown as follows:

$$\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T = ((0.513, 0.6875), (0.1834, 0.9272), (0.447, 0.9614), (0.5165, 0.6108))^T.$$

By Definition 2.3, $\mathscr{S}(\omega_1) = 0.4762$, $\mathscr{S}(\omega_2) = 0.3375$, $\mathscr{S}(\omega_3) = 0.32$, $\mathscr{S}(\omega_4) = 0.487$ and $\mathscr{S}(\omega_4) > \mathscr{S}(\omega_1) > \mathscr{S}(\omega_2) > \mathscr{S}(\omega_3)$. The ranking order of alternatives $a_4 > a_1 > a_2 > a_3$.

4 GDM with incomplete FFSs

This section is dedicated to defining a new GDM in the incomplete $\mathfrak{FR}$ environment and examining its key features.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be set of alternatives, and let $H = \{h_1, h_2, \ldots, h_m\}$ be a set of decision makers. Assume that $M^t = (m_{ij}^t)_{n\times n}$ is the $\mathfrak{FR}$ offered by decision maker h_t, where $(m_{ij}^t) = (\zeta_{ij}^t, \eta_{ij}^t)$ is the $\mathfrak{FFN}$ for $(i, j = 1, 2, \ldots, n;\ t = 1, 2, \ldots, m)$.

For the individual $\mathfrak{FR}$ $M^t = (m_{ij}^t)_{n\times n}$, where $(m_{ij}^t) = (\zeta_{ij}^t, \eta_{ij}^t)$, $i, j = 1, 2, \ldots, n$; $t = 1, 2, \ldots, m$, let $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\}^T$ be their weight vector, where ω_t denotes the weight of decision maker h_t, $\omega_t \in [0, 1]$, $t = 1, 2, \ldots, m$ and $\sum_{t=1}^{m}\omega_t = 1$. Then, the collective $\mathfrak{FR}$ $M^t = (m_{ij}^t)_{n\times n}$ is defined as follows:

$$m_{ij}^k = (\zeta_{ij}^k, \eta_{ij}^k) = \left(\left(\sum_{t=1}^{m}\omega_t(\zeta_{ij}^t)^3\right)^{1/3}, \left(\sum_{t=1}^{m}\omega_t(\eta_{ij}^t)^3\right)^{1/3}\right), \tag{14}$$

for all $i, j = 1, 2, \ldots, n$.

Note that

$$\zeta_{ij}^t = \eta_{ji}^t, \quad \zeta_{ji}^t = n_{ij}^t, \quad \zeta_{ii}^t = \eta_{ii}^t = 2^{-1/3}, \quad 0 \le (\zeta_{ij}^t)^3 + (\eta_{ij}^t)^3 \le 1, 0 \le \zeta_{ij}^t, \eta_{ij}^t \le 1.$$

for each pair of (i, j). Thus, we obtain,

$$\begin{cases} 0 \le \zeta_{ij}^k = \left(\sum_{t=1}^m \omega_t(\zeta_{ij}^t)^3\right)^{1/3} \le \left(\sum_{t=1}^m \omega_t 1^3\right)^{1/3} = 1, \\ 0 \le \eta_{ij}^k = \left(\sum_{t=1}^m \omega_t(\eta_{ij}^t)^3\right)^{1/3} \le \left(\sum_{t=1}^m \omega_t 1^3\right)^{1/3} = 1, \\ \zeta_{ii}^k = \left(\sum_{t=1}^m \omega_t(\zeta_{ii}^t)^3\right)^{1/3} = \left(\sum_{t=1}^m \omega_t(2^{-1/3})^3\right)^{1/3} = 2^{-1/3}, \\ \eta_{ii}^k = \left(\sum_{t=1}^m \omega_t(\eta_{ii}^t)^3\right)^{1/3} = \left(\sum_{t=1}^m \omega_t(2^{-1/3})^3\right)^{1/3} = 2^{-1/3}, \\ \zeta_{ij}^k = \left(\sum_{t=1}^m \omega_t(\zeta_{ij}^t)^3\right)^{1/3} = \left(\sum_{t=1}^m \omega_t(\eta_{ij}^t)^3\right)^{1/3} = \eta_{ij}^k, \\ \eta_{ij}^k = \left(\sum_{t=1}^m \omega_t(\eta_{ij}^t)^3\right)^{1/3} = \left(\sum_{t=1}^m \omega_t(\zeta_{ij}^t)^3\right)^{1/3} = \zeta_{ij}^k, \\ (\zeta_{ij}^t)^3 + (\eta_{ij}^t)^3 = \left(\left(\sum_{t=1}^m \omega_t(\zeta_{ij}^t)^3\right)^{1/3}\right)^3 + \left(\left(\sum_{t=1}^m \omega_t(\eta_{ij}^t)^3\right)^{1/3}\right)^3, \\ = \sum_{t=1}^m \omega_t(\zeta_{ij}^t)^3 + \sum_{t=1}^m \omega_t(\eta_{ij}^t)^3 = \sum_{t=1}^m \omega_t((\zeta_{ij}^t)^3 + (\eta_{ij}^t)^3) \le \sum_{t=1}^m \omega_t = 1. \end{cases}$$

Thus, M^k is a $\mathfrak{FR}$.

Theorem 4.1. *Let M^t be an $\mathfrak{FR}$, where $t = 1, 2, \ldots, m$. Then, their collective $\mathfrak{FR}$s M^t introduced in equation (14) is acceptable additive consistent if all $\mathfrak{FR}$s M^t, where $t = 1, 2, \ldots, m$ are acceptable additive consistent, namely, $\mathfrak{ACI}(M^t) \ge \alpha$ for all $t = 1, 2, \ldots, m$, with α being the given additive consistency threshold.*

Proof. As per equation (14) and the acceptable consistency of M^t, we have

$$\begin{aligned} \mathfrak{ACI}(M^t) &= 1 - (2/n(n-1)(n-2))\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} |(\zeta_{ij}^k)^3 + (\zeta_{jk}^k)^3 + (\eta_{ik}^k)^3 - (\eta_{ij}^k)^3 - (\eta_{jk}^k)^3 - (\zeta_{ik}^k)^3| \\ &= 1 - (2/n(n-1)(n-2))\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left| \left(\left(\sum_{t=1}^m \omega_t(\zeta_{ij}^t)^3\right)^{1/3}\right)^3 + \left(\left(\sum_{t=1}^m \omega_t(\zeta_{jk}^t)^3\right)^{1/3}\right)^3 \right. \\ &\quad \left. + \left(\left(\sum_{t=1}^m \omega_t(\eta_{ik}^t)^3\right)^{1/3}\right)^3 - \left(\left(\sum_{t=1}^m \omega_t(\eta_{ij}^t)^3\right)^{1/3}\right)^3 - \left(\left(\sum_{t=1}^m \omega_t(\eta_{jk}^t)^3\right)^{1/3}\right)^3 - \left(\left(\sum_{t=1}^m \omega_t(\zeta_{ik}^t)^3\right)^{1/3}\right)^3 \right| \\ &= 1 - (2/n(n-1)(n-2))\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left| \sum_{t=1}^m (\omega_t(\zeta_{ij}^t)^3) + \sum_{t=1}^m (\omega_t(\zeta_{jk}^t)^3) + \sum_{t=1}^m (\omega_t(\eta_{ik}^t)^3) \right. \\ &\quad \left. - \sum_{t=1}^m (\omega_t(\eta_{ij}^t)^3) - \sum_{t=1}^m (\omega_t(\eta_{jk}^t)^3) - \sum_{t=1}^m (\omega_t(\zeta_{ik}^t)^3) \right| \\ &= 1 - (2/n(n-1)(n-2))\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left| \sum_{t=1}^m (\omega_t((\zeta_{ij}^t)^3 + (\zeta_{jk}^t)^3 + (\eta_{ik}^t)^3 - (\eta_{ij}^t)^3 - (\eta_{jk}^t)^3 - (\zeta_{ik}^t)^3)) \right| \\ &\ge 1 - (2/n(n-1)(n-2))\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} \left(\omega_t \sum_{t=1}^m |(\zeta_{ij}^t)^3 + (\zeta_{jk}^t)^3 + (\eta_{ik}^t)^3 - (\eta_{ij}^t)^3 - (\eta_{jk}^t)^3 - (\zeta_{ik}^t)^3| \right) \\ &= \sum_{t=1}^m \left(\omega_t \left(1 - (2/n(n-1)(n-2))\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} |(\zeta_{ij}^t)^3 + (\zeta_{jk}^t)^3 + (\eta_{ik}^t)^3 - (\eta_{ij}^t)^3 - (\eta_{jk}^t)^3 - (\zeta_{ik}^t)^3| \right) \right) \end{aligned}$$

$$= \sum_{t=1}^{m} (\omega_t \mathfrak{AECI}(M^k))$$
$$\geq \sum_{t=1}^{m} (\omega_t \alpha) = \alpha.$$

Following Definition 2.11, it is concluded that M^t is acceptable additive consistent. □

According to the individual $\mathfrak{FR}$, and the collective $\mathfrak{FR}$, the consensus index is proposed as follows.

Definition 4.2. Let $M^t = (m_{ij}^t)_{n\times n}$ $(t = 1, 2, \ldots, m)$ be an $m\mathfrak{FR}$s, where $m_{ij}^t = (\zeta_{ij}^t, \eta_{ij}^t)$, and ω be their weight vector. Let $M^t = (m_{ij}^t)$ be the collective $\mathfrak{FR}$ of the $\mathfrak{FR}$s $M^1, M^2, \ldots, M^m$, where $m_{ij}^k = (\zeta_{ij}^k, \eta_{ij}^k)$. The consensus index $\mathfrak{CI}(M^t)$ is defined as follows:

$$\mathfrak{CI}(M^t) = 1 - (1 - (n(n-1)))\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} (|\zeta_{ij}^t - \zeta_{ij}^k| + |\eta_{ij}^t - \eta_{ij}^k|), \tag{15}$$

where $t = 1, 2, \ldots, m$.

Because $0 \leq |\zeta_{ij}^t - \zeta_{ij}^k|, |\eta_{ij}^t - \eta_{ij}^k| \leq 1$, we obtain $0 \leq \mathfrak{CI} \leq 1$ for each $t = 1, 2, \ldots, m$. Furthermore, $\mathfrak{CI}(M^t) = 1$ if and only if $M^t = M^u$ for all $t, u = 1, 2, \ldots, m$.

Let π be the given consensus threshold. To rank alternatives from acceptable additive consistent $\mathfrak{FR}$s M'^t $(t = 1, 2, \ldots, m)$ with the given consensus level, we offer the following model:

$$g = \min \sum_{s=1}^{m}\sum_{i}^{n-1}\sum_{j=i+1}^{n} (d_{ij}^{t-} + d_{ij}^{t+} + e_{ij}^{t-} + e_{ij}^{t+}),$$

$$\begin{cases} \mathfrak{CI}(M^t) \geq \pi, \quad t = 1, 2, \ldots, m, \\ \mathfrak{AECI}(M^t) \geq \alpha, \quad t = 1, 2, \ldots, m, \\ M'^t = (m_{ij}^t)_{n\times n}, \quad t = 1, 2, \ldots, m \\ m_{ij}^t = (\zeta_{ij}^t, \eta_{ij}^t) = (\zeta_{ij}^t + d_{ij}^{t-} - d_{ij}^{t+}, \eta_{ij}^t + e_{ij}^{t-} - e_{ij}^{t+}), \quad t = 1, 2, \ldots, m, \ i < j, \ i, j = 1, 2, \ldots, n, \\ M^k = (m_{ij}^k)n \times n, \\ m_{ij}^k = (\zeta_{ij}^k, \eta_{ij}^k) = \left(\left(\sum_{t=1}^{m} \omega_t (\zeta_{ij}^t)^3\right)^{1/3}, \left(\sum_{t=1}^{m} \omega_t (\eta_{ij}^t)^3\right)^{1/3}\right), \ i < j, \ i, j = 1, 2, \ldots, n, \\ 0 \leq \zeta_{ij}^t + d_{ij}^{t-} - d_{ij}^{t+} \leq 1, \quad t = 1, 2, \ldots, m, \ i < j, \ i, j = 1, 2, \ldots, n, \\ 0 \leq \eta_{ij}^t + e_{ij}^{t-} - e_{ij}^{t+} \leq 1, \quad t = 1, 2, \ldots, m, \ i < j, \ i, j = 1, 2, \ldots, n, \\ 0 \leq (\zeta_{ij}^t + d_{ij}^{t-} - d_{ij}^{t+})^3 + (\eta_{ij}^t + e_{ij}^{t-} - e_{ij}^{t+})^3 \leq 1, \quad t = 1, 2, \ldots, m, \ i < j, \ i, j = 1, 2, \ldots, n, \\ d_{ij}^{t-}, d_{ij}^{t+}, e_{ij}^{t-}, e_{ij}^{t+} \geq 0, \quad t = 1, 2, \ldots, m, \ i < j, \ i, j = 1, 2, \ldots, n, \end{cases} \tag{16}$$

where

$$d_{ij}^{t+} = \frac{|\zeta_{ij}^k - \zeta_{ij}'^k| + (\zeta_{ij}^k - \zeta_{ij}'^k)}{2}, \quad d_{ij}^{t-} = \frac{|\zeta_{ij}^k - \zeta_{ij}'^k| - (\zeta_{ij}^k - \zeta_{ij}'^k)}{2},$$
$$e_{ij}^{t+} = \frac{|\eta_{ij}^k - \eta_{ij}'^k| + (\eta_{ij}^k - \eta_{ij}'^k)}{2}, \quad e_{ij}^{t-} = \frac{|\eta_{ij}^k - \eta_{ij}'^k| - (\eta_{ij}^k - \eta_{ij}'^k)}{2},$$
$$|\zeta_{ij}^k - \zeta_{ij}'^k| = d_{ij}^{t+} + d_{ij}^{t-\prime} \quad \zeta_{ij}^k - \zeta_{ij}'^k = d_{ij}^{t+} - d_{ij}^{t-}$$
$$|\eta_{ij}^k - \eta_{ij}'^k| = e_{ij}^{t+} + e_{ij}^{t-\prime} \quad \eta_{ij}^k - \eta_{ij}'^k = e_{ij}^{t+} - e_{ij}^{t-}, \quad t = 1, 2, \ldots, m, \ i < j, \ i, j = 1, 2, \ldots, n.$$

To solve the model 16, we convert it into the following model by means of equations (1) and (15):

$$g = \min \sum_{s=1}^{m} \sum_{i}^{n-1} \sum_{j=i+1}^{n} (d_{ij}^{t-} + d_{ij}^{t+} + e_{ij}^{t-} + e_{ij}^{t+}),$$

$$\begin{cases} \sum\limits_{i}^{n-1} \sum\limits_{j=i+1}^{n} (|\zeta_{ij}^{\prime t} - \zeta_{ij}^{\prime k}| + |\eta_{ij}^{\prime t} - \eta_{ij}^{\prime k}|) \le n(n-1)(1-\pi), \quad t = 1, 2, \ldots, m, \\ \sum\limits_{i=1}^{n-2} \sum\limits_{i}^{n-1} \sum\limits_{j=i+1}^{n} |(\zeta_{ij}^{\prime t})^3 + (\zeta_{jk}^{\prime t})^3 + (\eta_{ik}^{\prime t})^3 - (\eta_{ij}^{\prime t})^3 - (\eta_{jk}^{\prime t})^3 - (\zeta_{ik}^{\prime t})^3| \le (n(n-1)(n-2)(1-\pi)/2), \\ t = 1, 2, \ldots, m, \\ M^{\prime t} = (m_{ij}^{\prime t})n \times n, \quad t = 1, 2, \ldots, m, \\ m_{ij}^{\prime t} = (\zeta_{ij}^{\prime t}, \eta_{ij}^{\prime t}) = (\zeta_{ij}^{t} + d_{ij}^{t-} - d_{ij}^{t+}, \eta_{ij}^{t} + e_{ij}^{t-} - e_{ij}^{t+}), \quad t = 1, 2, \ldots, m, \ i, j = 1, 2, \ldots, n, \quad i < j, \\ M^{\prime k} = (m_{ij}^{\prime k})n \times n, \\ m_{ij}^{\prime k} = (\zeta_{ij}^{\prime k}, \eta_{ij}^{\prime k}) = \left(\left(\sum\limits_{t=1}^{m} \omega_t (\zeta_{ij}^{\prime t})^3 \right)^{1/3}, \left(\sum\limits_{t=1}^{m} \omega_t (\eta_{ij}^{\prime t})^3 \right)^{1/3} \right), \quad i, j = 1, 2, \ldots, n, \quad i < j, \\ 0 \le \zeta_{ij}^{t} + d_{ij}^{t-} - d_{ij}^{t+} \le 1, \quad t = 1, 2, \ldots, m, \ i, j = 1, 2, \ldots, n, \quad i < j, \\ 0 \le \eta_{ij}^{t} + e_{ij}^{t-} - e_{ij}^{t+} \le 1, \quad t = 1, 2, \ldots, m, \ i, j = 1, 2, \ldots, n, \ i < j, \\ 0 \le (\zeta_{ij}^{t} + d_{ij}^{t-} - d_{ij}^{t+})^3 + (\eta_{ij}^{t} + e_{ij}^{t-} - e_{ij}^{t+})^3 \le 1, \quad t = 1, 2, \ldots, m, \ i, j = 1, 2, \ldots, n, \quad i < j, \\ d_{ij}^{t-}, d_{ij}^{t+}, e_{ij}^{t-}, e_{ij}^{t+} \ge 0, \quad t = 1, 2, \ldots, m, \ i, j = 1, 2, \ldots, n, \quad i < j, \end{cases} \tag{17}$$

where the first constraint is the consensus requirement, the second constraint is the acceptable additive consistency condition, and the remaining constraints ensure the modified $\mathfrak{FR}$s to be still $\mathfrak{FR}$s.

After solving the model 17, we can acquire the optimal solutions, denoted as d_{ij}^{*t-}, d_{ij}^{*t+}, e_{ij}^{*t-}, e_{ij}^{*t+}. Then, we can obtain the adjusted acceptable additive consistent $\mathfrak{FR}$s $M^{\prime t} = (m_{ij}^{\prime t})n \times n$ $(t = 1, 2, \ldots, m)$ with the given consensus level, where

$$m_{ij}^{\prime t} = (\zeta_{ij}^{\prime t}, \eta_{ij}^{\prime t}) = \begin{cases} (\zeta_{ij}^{t} + d_{ij}^{*t-} - d_{ij}^{*t+}, \eta_{ij}^{t} + e_{ij}^{*t-} - e_{ij}^{*t+}), & \text{if } i < j, \\ (2^{-1/3}, 2^{-1/3}), & \text{if } i = j, \\ (\eta_{ji}^{t} + t_{ji}^{*t-} - t_{ji}^{*t+}, \zeta_{ji}^{t} + s_{ji}^{*t-} - s_{ji}^{*t+}), & \text{if } i > j. \end{cases} \tag{18}$$

where $1 \le i \le n, \ 1 \le j \le n$.

We further offer the following model to adjust individual $\mathfrak{FR}$s:

$$v = \min \sum_{s=1}^{m} \sum_{i}^{n-1} \sum_{j=i+1}^{n} (y_{ij}^{t} + z_{ij}^{t}),$$

$$
\begin{cases}
\sum_{i}^{n-1}\sum_{j=i+1}^{n}(|\zeta_{ij}^{\prime t}-\zeta_{ij}^{\prime k}|+|\eta_{ij}^{\prime t}-\eta_{ij}^{\prime k}|)\le n(n-1)(1-\pi), \quad t=1,2,\ldots,m,\\
\sum_{i=1}^{n-2}\sum_{i}^{n-1}\sum_{j=i+1}^{n}|(\zeta_{ij}^{\prime t})^3+(\zeta_{jk}^{\prime t})^3+(\eta_{ik}^{\prime t})^3-(\eta_{ij}^{\prime t})^3-(\eta_{jk}^{\prime t})^3-(\zeta_{ik}^{\prime t})^3|\le\frac{n(n-1)(n-2)(1-\alpha)}{2},\\
t=1,2,\ldots,m,\\
M^{\prime t}=(m_{ij}^{\prime t})n\times n, \quad t=1,2,\ldots,m,\\
m_{ij}^{\prime t}=(\zeta_{ij}^{\prime t},\eta_{ij}^{\prime t})=(\zeta_{ij}^{t}+d_{ij}^{t-}-d_{ij}^{t+},\eta_{ij}^{t}+e_{ij}^{t-}-e_{ij}^{t+}),\ i,j=1,2,\ldots,n \quad i<j,\\
M^{\prime k}=(m_{ij}^{\prime k})n\times n,\\
m_{ij}^{\prime k}=(\zeta_{ij}^{\prime k},\eta_{ij}^{\prime k})=\left(\left(\sum_{t=1}^{m}\omega_t(\zeta_{ij}^{\prime t})^3\right)^{1/3},\left(\sum_{t=1}^{m}\omega_t(\eta_{ij}^{\prime t})^3\right)^{1/3}\right),\ i,j=1,2,\ldots,n \quad i<j,\\
\zeta_{ij}^{t}+d_{ij}^{t-}-d_{ij}^{t+}=(1-y_{ij}^{t})(\zeta_{ij}^{t}+d_{ij}^{t-}-d_{ij}^{t+})+y_{ij}^{t}\zeta_{ij}^{t},\ i,j=1,2,\ldots,n \quad i<j, \quad t=1,2,\ldots,m,\\
\eta_{ij}^{t}+e_{ij}^{t-}-e_{ij}^{t+}=(1-z_{ij}^{t})(\eta_{ij}^{t}+e_{ij}^{t-}-e_{ij}^{t+})+z_{ij}^{t}\eta_{ij}^{t},\\
y_{ij}^{t},z_{ij}^{t}\in\{0,1\},\ i,j=1,2,\ldots,n \quad i<j, \quad t=1,2,\ldots,m,\\
g^{*}=\sum_{i=1}^{n-2}\sum_{i}^{n-1}\sum_{j=i+1}^{n}(d_{ij}^{t-}+d_{ij}^{t+}+e_{ij}^{t-}+e_{ij}^{t+})\\
0\le\zeta_{ij}^{t}+d_{ij}^{t-}-d_{ij}^{t+}\le 1, \quad i,j=1,2,\ldots,n \quad i<j, \quad t=1,2,\ldots,m,\\
0\le\eta_{ij}^{t}+e_{ij}^{t-}-e_{ij}^{t+}\le 1, \quad i,j=1,2,\ldots,n \quad i<j, \quad t=1,2,\ldots,m,\\
0\le(\zeta_{ij}^{t}+d_{ij}^{t-}-d_{ij}^{t+})^3+(\eta_{ij}^{t}+e_{ij}^{t-}-e_{ij}^{t+})^3\le 1,\ i,j=1,2,\ldots,n \quad i<j, \quad t=1,2,\ldots,m,\\
d_{ij}^{t-},d_{ij}^{t+},e_{ij}^{t-},e_{ij}^{t+}\ge 0 \quad i,j=1,2,\ldots,n \quad i<j, \quad t=1,2,\ldots,m,
\end{cases}
\tag{19}
$$

where g^{*} is the optimal objective value of the Model 17 and also d_{ij}^{t-}, d_{ij}^{t+}, e_{ij}^{t-}, e_{ij}^{t+} are shown in the Model 17.

Solving the Model 19 derived its optimal solutions, denoted by $d_{ij}^{\diamond vt-}$, $d_{ij}^{\diamond t+}$, $e_{ij}^{\diamond t-}$, $e_{ij}^{\diamond t+}$. Thus, we can obtain the adjusted acceptable additive consistent $\mathfrak{FR}$ $M^{\prime t}=(m_{ij}^{\prime t})_{n\times n}$ with the given consensus level, where

$$
m_{ij}^{\prime t}=(\zeta_{ij}^{\prime t},\eta_{ij}^{\prime t})=\begin{cases}(\zeta_{ij}^{t}+d_{ij}^{\diamond t-}-d_{ij}^{\diamond t+},\eta_{ij}^{t}+e_{ij}^{\diamond t-}-e_{ij}^{\diamond t+}), & \text{if } i<j,\\ (2^{-1/3},2^{-1/3}), & \text{if } i=j,\\ (\eta_{ji}^{t}+e_{ji}^{\diamond t-}-e_{ji}^{\diamond t+},\zeta_{ji}^{t}+d_{ji}^{\diamond t-}-d_{ji}^{\diamond t+}), & \text{if } i>j,\end{cases}
\tag{20}
$$

where $1\le i\le n,\ 1\le j\le n$.

4.1 Algorithm

The weights of the decision makers are calculated with

$$
\omega_{h_t}=\frac{1/\left(\sum_{u=1;s\neq u}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\zeta_{ij}^{t}-\zeta_{ij}^{u}|+|\eta_{ij}^{t}-\eta_{ij}^{u}|)\right)}{\sum_{t=1}^{m}\left(1/\left(\sum_{u=1;s\neq u}^{m}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(|\zeta_{ij}^{t}-\zeta_{ij}^{u}|+|\eta_{ij}^{t}-\eta_{ij}^{u}|)\right)\right)}.
\tag{21}
$$

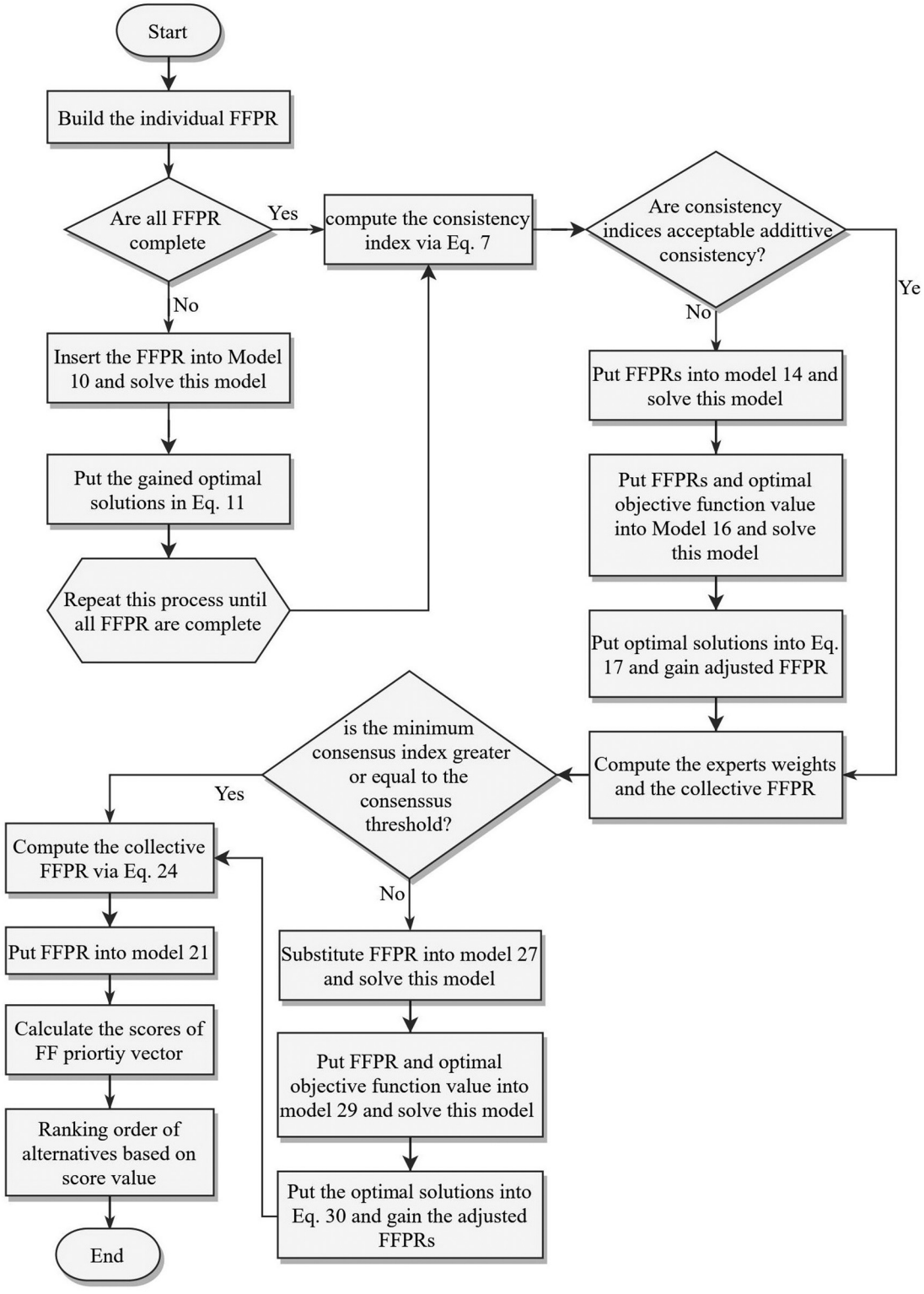


Figure 1: Flow charts for algorithms.

Algorithm 1 An algorithm to GDM with $\mathfrak{FR}$ (Figure 1)

Input: $\mathfrak{IFR}$ $M = (m_{ij}^t)_{n\times n}$ with missing values.
Output: The ranking order of the alternatives and choosing the best one of alternatives.
Step 1: Let $M = (m_{ij}^t)_{n\times n}$ be the individual $\mathfrak{FR}$ offered by the decision maker h_t. Are all $\mathfrak{FR}$s $M = (m_{ij}^t)_{n\times n}$ complete? If the answer is yes, go to Step 4.
Step 2: If all $\mathfrak{FR}$s $M = (m_{ij})_{n\times n}$ are incomplete then, insert M^t into Model 1. The next operation is to analyze this model.
Step 3: Using the acquired optimal solutions in equation (2), a complete $\mathfrak{FR}$ is found. Repeat this process until all $\mathfrak{FR}$s M^t are complete.
Step 4: Compute the consistency index $\mathfrak{CI}(M)$ of the $\mathfrak{FR}$ M^t via equation (2.9).
Step 5: If all of them are acceptable $\mathfrak{AC}$, go to Step 9.
Step 6: If all of them aren't acceptable $\mathfrak{AC}$, insert M^t with unacceptable additive consistent into the model 5. The optimal objective function value is acquired by solving this model.
Step 7: M^t and this optimal objective function value are put into the model 7. By analyzing this model, optimal solutions are acquired.
Step 8: Put these optimal solutions in equation (8). Adjusted $\mathfrak{FR}$ is achieved with acceptable additive consistency.
Step 9: The experts weights are calculated with equation (21). Using the equation (14), calculate the collective $\mathfrak{FR}$ M^t.
Step 10: Consider the preconcerted consensus threshold as ξ. When $\min\{\mathfrak{CI}(M)^a\} \geq \xi$, go to Step 13.
Step 11: Otherwise, substitute M^t into the Model 17. By solving this model, the optimal objective function value is acquired. M^t and this optimal objective function value are put into model 19. By solving this model, optimal solutions are obtained.
Step 12: Put these optimal solutions into equation (20) to obtain $\mathfrak{FR}$s adjusted with acceptable additive consistency and acceptable consensus.
Step 13: Compute the collective $\mathfrak{FR}$ M^t by aggregating $\mathfrak{FR}$s M^t via equation (14).
Inserting M^t into Model 13, a FF priority vector ω can be derived from M^t.
Step 14: Calculate the scores of ω_i. Based on the score value of ω_i, ranking order of alternatives.
Step 15: End.

5 Application

Let p_1, p_2, p_3, and p_4 be four experts and d_1, d_2, d_3, and d_4 denote the alternatives. Incomplete $\mathfrak{FR}$s M^1, M^2, M^3, and M^4, respectively, are given as follows:

$$M^1 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (-,-) & (0.82, 0.59) & (0.69, 0.92) \\ (-,-) & (2^{-1/3}, 2^{-1/3}) & (-,0.41) & (0.33, 0.52) \\ (0.59, 0.82) & (0.41, -) & (2^{-1/3}, 2^{-1/3}) & (0.69, -) \\ (0.92, 0.69) & (0.52, 0.33) & (-,0.69) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^2 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.48, 0.89) & (-,-) & (0.73, 0.20) \\ (0.89, 0.48) & (2^{-1/3}, 2^{-1/3}) & (0.71, 0.14) & (0.78, 0.27) \\ (-,-) & (0.14, 0.71) & (2^{-1/3}, 2^{-1/3}) & (-,-) \\ (0.2, 0.73) & (0.27, 0.78) & (-,-) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^3 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.61, 0.79) & (-,-) & (0.79, 0.56) \\ (0.79, 0.61) & (2^{-1/3}, 2^{-1/3}) & (-,0.36) & (0.56, 0.84) \\ (-,-) & (0.36, -) & (2^{-1/3}, 2^{-1/3}) & (0.77, 0.82) \\ (0.56, 0.79) & (0.84, 0.56) & (0.82, 0.77) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^4 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (-,-) & (0.65, -) & (0.79, 0.57) \\ (-,-) & (2^{-1/3}, 2^{-1/3}) & (0.81, 0.20) & (0.60, 0.80) \\ (-,0.65) & (0.20, 0.81) & (2^{-1/3}, 2^{-1/3}) & (-,-) \\ (0.57, 0.79) & (0.80, 0.60) & (-,-) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

In these incomplete $\mathfrak{FR}$s, an unknown or missing value was denoted by –. Putting the $\mathfrak{FR}$s M^t $t = 1, 2, 3, 4$ into the model 1. By solving this model, optimal solutions are obtained: For M^1, $\zeta_{12}^{*1} = 0.7712$, $\eta_{12}^{*1} = 0.8965$, $\zeta_{23}^{*1} = 0.8703$, $\zeta_{34}^{*1} = 0.9058$. For M^2, $\zeta_{13}^{*2} = 0.927$, $\eta_{13}^{*2} = 0.8034$, $\zeta_{34}^{*2} = 0.8271$, $\eta_{34}^{*2} = 0.7845$. For M^3, $\zeta_{13}^{*3} = 0.803$, $\eta_{13}^{*3} = 0.8168$, $\zeta_{23}^{*3} = 0.6481$. For M^4, $\zeta_{12}^{*4} = 0.8533$, $\eta_{12}^{*4} = 0.7503$, $\eta_{13}^{*4} = 0.001$, $\zeta_{34}^{*4} = 0.7789$, $\eta_{34}^{*4} = 0.9523$. If these optimal solutions are substituted into equation (2), then complete $\mathfrak{FR}$s are written as follows:

$$M^1 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.7712, 0.8965) & (0.82, 0.59) & (0.69, 0.92) \\ (0.8965, 0.7712) & (2^{-1/3}, 2^{-1/3}) & (0.8703, 0.41) & (0.33, 0.52) \\ (0.59, 0.82) & (0.41, 0.8703) & (2^{-1/3}, 2^{-1/3}) & (0.69, 0.9058) \\ (0.92, 0.69) & (0.52, 0.33) & (0.9058, 0.69) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^2 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.48, 0.89) & (0.927, 0.8034) & (0.73, 0.20) \\ (0.89, 0.48) & (2^{-1/3}, 2^{-1/3}) & (0.71, 0.14) & (0.78, 0.27) \\ (0.8034, 0.927) & (0.14, 0.71) & (2^{-1/3}, 2^{-1/3}) & (0.8271, 0.7845) \\ (0.2, 0.73) & (0.27, 0.78) & (0.7845, 0.8271) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^3 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.61, 0.79) & (0.803, 0.8168) & (0.79, 0.56) \\ (0.79, 0.61) & (2^{-1/3}, 2^{-1/3}) & (0.6481, 0.36) & (0.56, 0.84) \\ (0.8168, 0.803) & (0.36, 0.6481) & (2^{-1/3}, 2^{-1/3}) & (0.77, 0.82) \\ (0.56, 0.79) & (0.84, 0.56) & (0.82, 0.77) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^4 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.8533, 0.7503) & (0.65, 0.001) & (0.79, 0.57) \\ (0.7503, 0.8533) & (2^{-1/3}, 2^{-1/3}) & (0.81, 0.20) & (0.60, 0.80) \\ (0.001, 0.65) & (0.20, 0.81) & (2^{-1/3}, 2^{-1/3}) & (0.7789, 0.9523) \\ (0.57, 0.79) & (0.80, 0.60) & (0.9523, 0.7789) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

Based on equation (1), $\mathfrak{ACI}(M^1) = 0.93$, $\mathfrak{ACI}(M^2) = 0.83$, $\mathfrak{ACI}(M^3) = 0.85$, $\mathfrak{ACI}(M^4) = 0.88$. Choose $\alpha = 0.9$. All of $\mathfrak{FR}$s are acceptable $\mathfrak{AC}$ except for the $\mathfrak{FR}$ M^2. Put M^2 into the model 5. By solving this model, the optimal objective function value of $f^* = 0.198$ is obtained. Hence, put M^2 and f^* into the model 7. By solving this model, optimal solutions are obtained as follows:

$$\begin{aligned}
&s_{12}^{\diamond-} = 0,\ s_{13}^{\diamond-} = 0,\ s_{14}^{\diamond-} = 0.001,\ s_{23}^{\diamond-} = 0,\ s_{24}^{\diamond-} = 0.001,\ s_{34}^{\diamond-} = 0,\ s_{12}^{\diamond+} = 0,\ s_{13}^{\diamond+} = 0,\\
&s_{14}^{\diamond+} = 0,\ s_{23}^{\diamond+} = 0.014,\ s_{24}^{\diamond+} = 0,\ s_{34}^{\diamond+} = 0,\ t_{12}^{\diamond-} = 0,\ t_{13}^{\diamond-} = 0,\ t_{14}^{\diamond-} = 0,\\
&t_{23}^{\diamond-} = 0,\ t_{24}^{\diamond-} = 0,\ t_{34}^{\diamond-} = 0,\ t_{12}^{\diamond+} = 0,\ t_{13}^{\diamond+} = 0,\ t_{14}^{\diamond+} = 0.12,\ t_{23}^{\diamond+} = 0,\ t_{24}^{\diamond+} = 0.17,\\
&t_{34}^{\diamond+} = 0,\ u_{123}^{\diamond-} = 0,\ u_{124}^{\diamond-} = 0.59,\ u_{134}^{\diamond-} = 0,\ u_{134}^{\diamond-} = 0,\ u_{234}^{\diamond-} = 0,\ u_{145}^{\diamond-} = 0,\\
&u_{123}^{\diamond+} = 0,\ u_{124}^{\diamond+} = 0,\ u_{134}^{\diamond+} = 0,\ u_{234}^{\diamond+} = 0.59,\\
&v_{12}^{\diamond} = 1,\ v_{13}^{\diamond} = 1,\ v_{14}^{\diamond} = 1,\ v_{24}^{\diamond} = 1,\ v_{34}^{\diamond} = 1,\\
&y_{12}^{\diamond} = 1,\ y_{13}^{\diamond} = 1,\ y_{14}^{\diamond} = 1,\ y_{23}^{\diamond} = 1,\ y_{24}^{\diamond} = 0,\ y_{34}^{\diamond} = 1.
\end{aligned}$$

We substitute these optimal solutions into equation (8) to obtain $\mathfrak{FR}$ M^2 adjusted with $\mathfrak{AC}$. Here,

$$M^2 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.48, 0.89) & (0.927, 0.8034) & (0.74, 0.08) \\ (0.89, 0.48) & (2^{-1/3}, 2^{-1/3}) & (0.71, 0.126) & (0.781, 0.1) \\ (0.8034, 0.927) & (0.126, 0.71) & (2^{-1/3}, 2^{-1/3}) & (0.8271, 0.7845) \\ (0.08, 0.74) & (0.1, 0.781) & (0.7845, 0.8271) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

By using equation (21), it is calculated as $\omega = \{0.272, 0.217, 0.282, 0.23\}^T$ for experts. The collective $\mathfrak{FR}$ M^k is acquired using equation (14):

$$M^k = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.727, 0.797) & (0.838, 0.273) & (0.696, 0.425) \\ (0.797, 0.737) & (2^{-1/3}, 2^{-1/3}) & (0.751, 0.297) & (0.752, 0.462) \\ (0.273, 0.838) & (0.297, 0.751) & (2^{-1/3}, 2^{-1/3}) & (0.774, 0.866) \\ (0.425, 0.696) & (0.462, 0.752) & (0.866, 0.774) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

By using equation (15), we find $\mathfrak{CI}(M^1) = 0.814$, $\mathfrak{CI}(M^2) = 0.815$, $\mathfrak{CI}(M^3) = 0.825$, $\mathfrak{CI}(M^4) = 0.862$. Choose the consensus threshold as $\pi = 0.9$. Substitute M^t $(t = 1, 2, 3, 4)$ in the model 17 to obtain the optimal objective function value. Therefore, $g^* = 1.8327$. Hence, put the M^t $(t = 1, 2, 3, 4)$ and $g^* = 1.8327$ into the model 19. To acquire optimal solutions, the model must be analyzed:

$d_{12}^{\diamond 1-} = 0$, $d_{13}^{\diamond 1-} = 0$, $d_{14}^{\diamond 1-} = 0$, $d_{23}^{\diamond 1-} = 0$, $d_{24}^{\diamond 1-} = 0$, $d_{34}^{\diamond 1-} = 0$, $d_{12}^{\diamond 1+} = 0$, $d_{13}^{\diamond 1+} = 0.016$,
$d_{14}^{\diamond 1+} = 0$, $d_{23}^{\diamond 1+} = 0$, $d_{24}^{\diamond 1+} = 0$, $d_{34}^{\diamond 1+} = 0$, $e_{12}^{\diamond 1-} = 0$, $e_{13}^{\diamond 1-} = 0$, $e_{14}^{\diamond 1-} = 0$, $e_{23}^{\diamond 1-} = 0$,
$e_{24}^{\diamond 1-} = 0$, $e_{34}^{\diamond 1-} = 0$, $e_{12}^{\diamond 1+} = 0$, $e_{13}^{\diamond 1+} = 0$, $e_{14}^{\diamond 1+} = 0.12$, $e_{23}^{\diamond 1+} = 0.039$, $e_{24}^{\diamond 1+} = 0$, $e_{34}^{\diamond 1+} = 0$
$d_{12}^{\diamond 2-} = 0.2176$, $d_{13}^{\diamond 2-} = 0$, $d_{14}^{\diamond 2-} = 0.001$, $d_{23}^{\diamond 2-} = 0$, $d_{24}^{\diamond 2-} = 0$, $d_{34}^{\diamond 2-} = 0$, $d_{12}^{\diamond 2+} = 0$, $d_{13}^{\diamond 2+} = 0$,
$d_{14}^{\diamond 2+} = 0$, $d_{23}^{\diamond 2+} = 0$, $d_{24}^{\diamond 2+} = 0.0394$, $d_{34}^{\diamond 2+} = 0.1338$, $e_{12}^{\diamond 2-} = 0$, $e_{13}^{\diamond 2-} = 0$, $e_{14}^{\diamond 2-} = 0.0066$,
$e_{23}^{\diamond 2-} = 0.0123$, $e_{24}^{\diamond 2-} = 0.5097$, $e_{34}^{\diamond 2-} = 0$, $e_{12}^{\diamond 2+} = 0.0001$, $e_{13}^{\diamond 2+} = 0.1247$, $e_{14}^{\diamond 2+} = 0$, $e_{23}^{\diamond 2+} = 0$, $e_{24}^{\diamond 2+} = 0$,
$e_{34}^{\diamond 2+} = 0$, $d_{12}^{\diamond 3-} = 0$, $d_{13}^{\diamond 3-} = 0$, $d_{14}^{\diamond 3-} = 0$, $d_{23}^{\diamond 3-} = 0$, $d_{24}^{\diamond 3-} = 0$, $d_{34}^{\diamond 3-} = 0$, $d_{12}^{\diamond 3+} = 0$, $d_{13}^{\diamond 3+} = 0$,
$d_{14}^{\diamond 3+} = 0$, $d_{23}^{\diamond 3+} = 0$, $d_{24}^{\diamond 3+} = 0$, $d_{34}^{\diamond 3+} = 0$, $e_{12}^{\diamond 3-} = 0$, $e_{13}^{\diamond 3-} = 0$, $e_{14}^{\diamond 3-} = 0$, $e_{23}^{\diamond 3-} = 0$,
$e_{24}^{\diamond 3-} = 0$, $e_{34}^{\diamond 3-} = 0$, $e_{12}^{\diamond 3+} = 0$, $e_{13}^{\diamond 3+} = 0$, $e_{14}^{\diamond 3+} = 0$, $e_{23}^{\diamond 3+} = 0$, $e_{24}^{\diamond 3+} = 0$, $e_{34}^{\diamond 3+} = 0$
$d_{12}^{\diamond 4-} = 0$, $d_{13}^{\diamond 4-} = 0.0957$, $d_{14}^{\diamond 4-} = 0$, $d_{23}^{\diamond 4-} = 0$, $d_{24}^{\diamond 4-} = 0.0029$, $d_{34}^{\diamond 4-} = 0$, $d_{12}^{\diamond 4+} = 0$, $d_{13}^{\diamond 4+} = 0.016$,
$d_{14}^{\diamond 4+} = 0$, $d_{23}^{\diamond 4+} = 0$, $d_{24}^{\diamond 4+} = 0$, $d_{34}^{\diamond 4+} = 0$, $e_{12}^{\diamond 4-} = 0$, $e_{13}^{\diamond 4-} = 0.1975$, $e_{14}^{\diamond 4-} = 0$, $e_{23}^{\diamond 4-} = 0$,
$e_{24}^{\diamond 4-} = 0$, $e_{34}^{\diamond 4-} = 0$, $e_{12}^{\diamond 4+} = 0$, $e_{13}^{\diamond 4+} = 0$, $e_{14}^{\diamond 4+} = 0.0089$, $e_{23}^{\diamond 4+} = 0$, $e_{24}^{\diamond 4+} = 0.1994$, $e_{34}^{\diamond 4+} = 0$.

Furthermore, add the aforementioned solutions to equation (20). Get acceptable $\mathfrak{AC}$ and four adjusted $\mathfrak{FR}$s by consensus. Here,

$$M^1 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.7712, 0.8965) & (0.804, 0.59) & (0.69, 0.8) \\ (0.8965, 0.7712) & (2^{-1/3}, 2^{-1/3}) & (0.8703, 0.371) & (0.33, 0.52) \\ (0.59, 0.804) & (0.371, 0.8703) & (2^{-1/3}, 2^{-1/3}) & (0.69, 0.9058) \\ (0.8, 0.69) & (0.52, 0.33) & (0.9058, 0.69) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^2 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.6976, 0.8899) & (0.927, 0.6787) & (0.741, 0.0866) \\ (0.8899, 0.6976) & (2^{-1/3}, 2^{-1/3}) & (0.71, 0.1383) & (0.7416, 0.6097) \\ (0.6787, 0.927) & (0.1383, 0.71) & (2^{-1/3}, 2^{-1/3}) & (0.6933, 0.7845) \\ (0.0866, 0.741) & (0.6097, 0.7416) & (0.7845, 0.6933) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^3 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.61, 0.79) & (0.803, 0.8168) & (0.79, 0.56) \\ (0.79, 0.61) & (2^{-1/3}, 2^{-1/3}) & (0.6481, 0.36) & (0.56, 0.84) \\ (0.8168, 0.803) & (0.36, 0.6481) & (2^{-1/3}, 2^{-1/3}) & (0.77, 0.82) \\ (0.56, 0.79) & (0.84, 0.56) & (0.82, 0.77) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}$$

$$M^4 = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.8533, 0.7503) & (0.7297, 0.001) & (0.79, 0.5611) \\ (0.7503, 0.8533) & (2^{-1/3}, 2^{-1/3}) & (0.81, 0.20) & (0.6029, 0.6006) \\ (0.001, 0.7297) & (0.20, 0.81) & (2^{-1/3}, 2^{-1/3}) & (0.7789, 0.9523) \\ (0.5611, 0.79) & (0.6006, 0.6029) & (0.9523, 0.7789) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

Using equation (21), with respect to experts, $\omega = \{0.262, 0.257, 0.243, 0.238\}^T$ is calculated. Using equation (14), the collective $\mathfrak{FR}\ M^k$ is obtained as follows:

$$M^k = \begin{bmatrix} (2^{-1/3}, 2^{-1/3}) & (0.754, 0.83) & (0.741, 0.151) & (0.752, 0.379) \\ (0.83, 0.754) & (2^{-1/3}, 2^{-1/3}) & (0.836, 0.261) & (0.606, 0.635) \\ (0.151, 0.741) & (0.261, 0.836) & (2^{-1/3}, 2^{-1/3}) & (0.736, 0.836) \\ (0.379, 0.752) & (0.635, 0.606) & (0.836, 0.736) & (2^{-1/3}, 2^{-1/3}) \end{bmatrix}.$$

When M^k is put into the model 13 and this model is solved, the four priority weights of the alternatives are calculated as $\omega_1 = (0.6196, 0.8043)$, $\omega_2 = (0.001, 0.7361)$, $\omega_3 = (0.001, 0.8965)$, and $\omega_4 = (0.5804, 0.7648)$.

Using equation (1), $\mathscr{S}(\omega_1) = 0.457$, $\mathscr{S}(\omega_2) = 0.3$, $\mathscr{S}(\omega_3) = 0.123$, $\mathscr{S}(\omega_4) = 0.392$. Since $\mathscr{S}(\omega_1) > \mathscr{S}(\omega_4) > \mathscr{S}(\omega_2) > \mathscr{S}(\omega_3)$, the ranking order $x_1 > x_4 > x_2 > x_3$. So, the best alternative is x_1.

6 Discussion

6.1 Comparative analysis

We consider that some unknown elements exist in incomplete FFPRs in the proposed GDM algorithm. Our proposed estimation model can reasonably estimate unknown elements in incomplete q-ROFPRs. However, the methods for GDM presented in [3,29,35,42] can only deal with complete PRs. As a result, our proposed method outperforms the methods presented in [19,46,47,52,53] for dealing with GDM problems because our GDM method can deal with incomplete FFPRs, whereas the methods presented in [19,46,47,52,53] for GDM cannot.

The weights of DMs are predefined in the GDM process of the methods presented in [19,46,47,52,53]. The proposed algorithm's GDM process employs the formula shown in equation (17) to generate DM weights that take into account the group consensus level. As a result, our algorithm is more rational than the GDM methods presented in [19,46,47,52,53].

The methods presented in [19,46,47,52,53] only define consistency; they do not identify the level of consistency and do not provide a consistency improvement approach for GDM. The proposed GDM algorithm includes a consistency adjustment mechanism that includes both a consistency check process and a consistency modification approach. As a result, the proposed algorithm is more practical than the methods presented in [19,46,47,52,53].

The method presented in [19,46,47,52,53] does not check for and achieve acceptable PR consensus. The proposed algorithm for GDM, on the other hand, considers consistency and consensus simultaneously, ensuring the minimum total adjustment and the minimum number of adjustments in FPRs, and ensuring reliable and effective results.

6.2 Superiority of new method

Making judgments regarding real-world situations, or finding solutions to them, is challenging and complex. Therefore, it is crucial to reduce uncertainty while deciding to select the best option or alternatives. A

generalization of the classical set, which consists of the FS, IFS, and PFS, is the FFS. As one of the most popular extensions, PFS is distinguished by the levels of membership and nonmembership satisfaction of the specific option within the criteria, where their total squares are equal to or less than 1. There may be instances where the decision-maker provides the degree of membership and nonmembership of a certain characteristic in a way that causes the total of the square to be more than 1. As a result, the PFS does not appropriately manage this case. To overcome this weakness, FFS theory is one of the more broad ones that can handle not just incomplete information but also indeterminate and inconsistent information, which are both frequently present in actual circumstances. Therefore, genuine scientific and engineering applications are better suited for Fermatean fuzzy information decision-making.

Our new method uses incomplete PRs, while the majority of previously given methods use complete PRs. While consistency and consensus detections are not considered in many previous methods, these detections are considered in the method we recommend. The method we propose calculates the weights of decision makers with equation (21). Therefore, more realistic results are achieved compared to predefined weights. In this study, a new score function is given. This score function is much more sensitive than the standard score function defined for FFS. This score function is used in our proposed method and provides a more precise measurement for ranking. Since FFS are generalizations of IFSs and PFSs, newly defined incomplete FFS also give better results in decision-making problems.

7 Conclusion

In this study, $\mathfrak{FR}$ was defined for GDM. In general, PRs are a considerable instrument in DM processes that can express preference flexibility of choice by the actual situation. Additive and multiplicative consistencies were widely used to estimate missing preference values across all preference relations. Implementing additive consistency to estimate missing preference values may not always be sufficient. To solve this matter, a new approach to GDM problems with missing $\mathfrak{FR}$s is presented. The additive consistency in the proposed technique has been defined by an FF priority vector. This model has been proposed to obtain missing judgments in incomplete $\mathfrak{FR}$s. To adjust the inconsistency of FFPRs, a method is given to increase consensus degrees with the presented procedure. The proposed method is very useful for GDM in incomplete $\mathfrak{FR}$ environments.

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