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Analytical Investigation of Nonlinear Distribution Mechanisms in Fluid Dynamics through the Boiti-Leon-Manna-Pempinelli Equation

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Abstract
The 3 + 1-dimensional Boiti-Leon-Manna-Pempinelli equation, a well-known higher-dimensional extension of the shallow-water wave model, has been taken into account. This article focuses on exploring the nonlinear dispersion mechanism by considering the wave velocity factor to analyze the behavior of incompressible fluids. To do this, two analytical methodologies are suggested, and solutions are generated that simulate many varieties of traveling waves under particular constraining circumstances. The stability of the equation considered in the study has beenevaluated using a linear stability analysis method that examines the effects of small-amplitude perturbations on the solution. The nonlinear wave distribution in the x direction is examined by selecting one of these solutions, which is the model of the time-dependent behavior of incompressible fluids, and by assigning the proper values to the constants that keep the system stable. In addition, while comparing the advantages and limitations of the two analytical methods, the study also draws attention to the strong physical correlation between diffusion, absorption, and wave velocity. Since wave velocity is scientifically related to energy per unit space or time, it has been concluded that high-energy waves propagate and diffuse more rapidly. The steady-state stability of the equation, which is the model of the nonlinear distribution, is determined by stimulated Raman scattering, group velocity distribution, and eigen-phase modulation. It has been argued that fluid flow modeling, simulation, and use of analytical results can help throughout the design phase to produce safe and useful products.

Keywords The modified $\left(\frac{1}{G'}\right)$-expansion technique · The 3 + 1-dimensional Boiti–Leon–Manna–Pempinelli equation · The $\left(\frac{G'}{G^2}\right)$-expansion technique · Exact wave solutions · Behavior of incompressible fluids

Introduction

Many scientists have been actively working in academia to seek accurate solutions of nonlinear evolution models in disciplines like fluid mechanics, meteorology, optics, plasma and solid state physics. [1, 2]. These soliton solutions can be used to represent various phenomena in chemistry, physics and other disciplines and provide additional information about the mechanism of systems. [3, 4]. Scientists have recently focused their efforts on equations 1 + 1 and 2 + 1 dimensional non-linear evolution models [5, 6]. However, real physical spacetime is a 3 + 1-dimensional equation, so nonlinear equations with more than two space dimensions are of greater interest [7, 8]. High-dimensional nonlinear evolution equations (NLEEs) include more accurate solutions, such as storm waves, respiratory waves, and periodic waves, than do low-dimensional models [9,



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10]. Numerous researchers have used the principle of linear superposition to obtain important results [11, 12]. The hyperbolic and trigonometric functions derived under certain conditions by applying the principle of linear superposition provided information about the state of traveling waves and were later extended and used to generate complex exponential wave function solutions [13, 14]. It is well known that there are various solutions for NPDEs, especially soliton equations in mathematical physics, such as rational, periodic, soliton, and complex [15, 16]. These solutions are an important instrument for scientific discussions and different perspectives on nonlinear scientific phenomena such as physics, plasma physics, chemistry and fluid dynamics [17, 18]. Finding the function that accurately represents the traveling wave under carefully selected limiting or beginning circumstances is typically the goal of the analytical solution of partial differential equations. These solution help us comprehend practical issues more fully, forecast future actions, and learn more about the relevant phenomenon. Many techniques are used to produce such solutions. For example modified generalized exponential rational function technique [19], the BNN approach [20], new auxiliary equation [21], the φ^6-Model expansion approach [22, 23], $\left(\frac{G'}{G}-\frac{1}{G}\right)$-expansion technique [24], $\left(\frac{1}{G'}\right)$-expansion technique [25, 26], Jacobi elliptic expansion function method [27], the unified method [28], Hirota bilinear method [29–31], Laplace transform collocation method [32], Lie symmetries [33], Hermite collocation method [34], new form of modified Kudryashov's method [35]. In applied science research, the investigation of integrable properties and precise outcomes of nonlinear partial differential problems has received a lot of interest lately [36, 37]. Recent progress has been made in research study on the quantitative as well as qualitative properties of the models. In this context, equations with variable coefficients have attracted particular attention. One of the most important reasons for this is that physical phenomena are worth explaining more realistically with variable coefficient NLDEs [38, 39].

Incompressible fluids are liquids that cannot be compressed. When pressure is applied to liquids, the volume or density does not change quickly. For a fluid to be incompressible, the ratio between flow velocity and sound velocity in the medium must be less than 0.3, according to fluid dynamics. As a result, for liquids, this ratio is smaller than 0.3, indicating that it is an incompressible fluid. Boiti–Leon–Manna–Pempinelli (BLMP) model is one of the most studied models in incompressible fluid [40]. Wazwaz and Mehanna used some techniques to generate traveling wave solutions [41] with the help of exp function and tanh-coth approaches, and they contrasted the two ways. Ansätz functions were first used by Long Wei [42] using the space-time three dimensional Kadomtsev–Petviashvili model. Later, this approach was extended to the space-time four dimensional high-order differential model, and infinite symmetry and conservation laws were examined. The 2 + 1-BLMP model

$$u_{yt} - 3u_{xy}u_x + u_{xxxy} - 3u_{xx}u_y = 0, \tag{1}$$

Gilson et al. [43] derived this model, which has recently been explored by a number of scientists. This equation is used are used to describe the 2 + 1-dimensional behavior of a Riemann wave traveling down the y-axis with a long wave moving along the x-axis. Darvishi et al. [44] proposed the 3 + 1-dimensional BLMP model as an improvement of the 2 + 1-dimensional equation, which models the connection of the Riemann wave along the y-axis with a long wave propagated down the x-axis of the 2 + 1 model.

$$u_{yt} - 3u_x(u_{xy} + u_{xz}) - 3u_{xx}\left(u_y + u_z\right) + u_{xxxy} + u_{zt} + u_{xxxz} = 0, \tag{2}$$

As an alternate version of the KdV equation, Boiti, Pempinelli and Leon generated the preliminary structure for this system. In other words, arbitrary functions of time and space were created as a novel version of the existing KdV model. Equation (2) is one of the few known integrable equations in 3 + 1 dimensions. The shallow water wave model is expanded to higher dimensions by this equation, which makes it extremely significant. It offers information about how ion acoustic waves behave in an incompressible manner. The dynamics of waves in the stratified ocean interior are likewise explained by the equation. It advances our knowledge of

wave mechanics in intricate systems by tackling these occurrences. This makes it an important tool for studying interactions between waves in several dimensions. Additionally, this equation is a more general spatial dimensional extension of classical shallow water wave models, incorporating nonlinear evolution, dispersion, and higher-order interaction terms. This structure makes the equation suitable for modeling the dynamic behavior of incompressible fluids. The (3+1)-dimensional form allows for the analysis of wave propagation in different spatial directions, enabling the examination of more complex physical situations. The main reason for choosing this model is that its high-dimensional nonlinear structure produces mathematically rich solution profiles and enables realistic modeling of physical processes. In addition, since the equation contains a large number of parameters, various physical scenarios can be simulated by selecting the appropriate parameters. For example, the physical relationship between the increase in wave energy and the increase in wave velocity has been directly analyzed using this model.

Multi-wave solutions for the BLMP equation were obtained by Darvishi [44]. A multiple expression approach was used to get the solutions. The technique made it easier to investigate intricate wave interactions inside the model. The results advance our knowledge of the dynamic characteristics of the equation. The efficiency of the multiple expression approach in resolving these kinds of equations is demonstrated by Darvishi's work. Completely novel bi-periodic soliton solutions were found using the extended homoclinic test approach, the bilinear form, and ansätz functions. A new class of wave events in nonlinear equations is represented by these solutions. To comprehend how these bi-periodic solitons behaved in various situations, their characteristics were carefully investigated. The physical structures connected to these solutions were also thoroughly examined. Significant new information about the characteristics of the solitons and their interactions was uncovered by the investigation. These results advance our knowledge of soliton solutions in nonlinear systems. References [45, 46] give the findings and in-depth analyses. Kinky periodic solitary-wave solutions were among the wave solutions that were obtained for this particular issue. Additionally, as part of the research, periodic soliton solutions were produced. The behavior of the equation under various circumstances may be understood from these solutions [46]. The Wronskian approach was used to obtained some exact solutions which were found in the form of rational solutions, soliton solutions [47]. Lax pairs, multiple soliton solutions were found using Backlund transformations, and a double linear form was obtained by using a logarithm transformation for the equation [48]. Rational and trigonometric wave solutions were successfully retrieved with the help of the Riemann theta function, ansatz and Pfaffian techniques [49]. Jiangen Liu et al. [50] applied the linear superposition approach and obtained complex solutions to create a special type of N-soliton solutions that are linear combinations of exponentially traveling waves for the $3+1$-dimensional BLMP equation. G. Xu [51] investigated the (3+1) dimensional BLMP model by using variable separation approach, Painlevée analysis and Hirota's bilinear approach, and obtained solutions that consist of physically N bending waves. Through the use of the extended homoclinic test approach, ansatz functions, and the Bilinear form, Jian-Guo Liu [52] presented completely new bi-periodic solutions to the model. In particular, these techniques were used to study incompressible fluid dynamics. The method was a major step forward in obtaining such results. Utilizing these mathematical resources, Liu made significant progress in resolving intricate fluid equations. Tolga Akturk [53] has successfully obtained the new moving wave solutions to the BLMP model by using the modified exponential function approach.

In this study, a new approach has been developed to the (3+1) dimensional BLMP model. Using the $\left(\frac{1}{G'}\right)$-expansion approach and $\left(\frac{G'}{G^2}\right)$-expansion approach, new traveling wave solutions of the BLMP model have been found. These methods have been chosen because they are effective, systematic, and applicable in nonlinear high-dimensional differential equations. The novelty in selecting these methods lies in the fact that they have not been applied together and comparatively on the same equation. Thus, the diversity of solutions produced by the methods, their parametric behavior, and their physical meanings have been analyzed. These approaches not only provide new solutions but also offer the opportunity to comparatively evaluate the analytical power of the methods. In order to obtain nonlineer analytical solutions of the equation, traveling wave solutions were found by choosing appropriate constants, and these solutions were expressed as trigonometric and hyperbolic functions.

However, the applicability of these methods is limited under certain mathematical and physical conditions. In particular, each solution type requires different preconditions depending on the structural characteristics of the method used. This situation limits the validity of the solution within certain boundaries and creates different limitations for each solution. Some solutions only have physical meaning under fixed parameters, while others are valid for solitons and periodic wave types. Even small changes in the parameters can affect the structure and physical interpretation of the solution.

The originality of the study is not limited to obtaining analytical solutions to the equation, but is also strengthened by revealing the physical relationship between diffusion, absorption, and wave velocity through these solutions. In particular, analyses conducted through the connection between wave velocity and energy density have shown that the solutions obtained within certain parameter ranges represent high-energy and fast-moving wave structures. Additionally, two- and three-dimensional graphical representations of these solutions have been created, and the soliton solution of the model will be examined in the study. Understanding the distinctive features of the soliton solution and the guiding ideas it offers constitutes the main focus of the analysis. This will enable a comprehensive evaluation of the model's applicability and physical significance.

This work is organized as follows: we begin with a quick overview of the $\left(\frac{G'}{G^2}\right)$-expansion technique and $\left(\frac{1}{G'}\right)$-expansion technique in the next section. We use the $\left(\frac{1}{G'}\right)$-expansion technique on the (3+1)-dimensional BLMP equations in section 3, and the $\left(\frac{G'}{G^2}\right)$-expansion technique in section 4 to derive travelling-wave solutions for our equation in these parts, which also includes graphs of the acquired results. In section 5, the stability analysis of the model is given. Section 6 contains the result and explanation of the acquired result, while section 6 concludes the work.

Fundamental concepts of the proposed method

Concepts of the modified $\left(\frac{1}{G'}\right)$-expansion technique

The general features of the modified (1/*G*)-expansion technique are presented in this section. To begin, we will look at a general form of nonlinear evolution equation that is dependent on x and t variables.

$$Q(u, u_t, u_x, u_{xx}, ...) = 0, \tag{3}$$

$u(t,x,y,z)$ is a function that depends on z,y,x and t, and $\xi = z + y + x - ct$ in the form $u(t, x, y, z) = u(\xi)$. Although c is mathematically a constant, it physically represents the velocity of the traveling wave. The PDF becomes an ordinary differential equation that is nonlinear as a result of this modification. This approach results in a clear nonlinear structure for solving it. This reformation makes it possible to use more focused computational and analytical methods.

$$W(u', u'', u''', ...) = 0. \tag{4}$$

The formula for the proposed solution to the nonlinear ordinary differential equation is $G = G(\xi)$. This illustration emphasizes how dependent G is on the parameter ξ. The solution is given below in the most straightforward form.

$$G'' + \lambda G' + \mu = 0, \tag{5}$$

which has the following solution

$$G' = -\frac{\mu}{\lambda} - \psi \sinh(\xi\lambda) + \psi \cosh(\xi\lambda), \tag{6}$$

In this case, ψ stands for the integration constant. A framework for expressing the solution to the nonlinear ordinary differential equation is given by equation (4). One of the most important steps in determining the solution is this equation. It makes it possible to formulate the solution precisely and effectively.

$$u(\xi) = \sum_{i=1}^{N} \left(a_i \left(\frac{1}{G'} \right)^m + a_{-i} \left(\frac{1}{G'} \right)^{-m} \right), \tag{7}$$

here m is a balancing term, while a_1, a_2, a_3, a_4 ,..., a_m are arbitrary constants to be determined later, the balancing term refers to the number generated by balancing any nonlinear ODE with the highest order linear term. The obtained number is now written in Equation (7), and then the solution's required derivatives are generated. By putting $\left(\frac{1}{G'}\right)$ into equation (5) a polynomial and homogeneous equation in such derivatives is obtained. Setting the coefficients of the variables $\left(\frac{1}{G'}\right)^m, m \in Q$, to zero, the system of algebraic equations is solved using a computer program, the obtained solution of the variables are substituted in equation (4) together with equation (7). The wave transformation is reversed once the proper controls are applied, and we have the solution of equation (3).

Concepts of the $\left(\frac{G'}{G^2}\right)$-expansion technique

For the purpose of obtaining traveling wave solutions to nonlinear partial differential models, expansion techniques are crucial mathematical instruments. Simplifying a partial differential equation into an ordinary differential equation is a crucial capability that both approaches have in common. To accomplish this reduction, a traditional wave transformation is used. These methods give complicated nonlinear systems an effective foundation for analysis. A key factor in making the equations' mathematical structure simpler is the transformation procedure. Through the reduction of dimensionality, these techniques allow for a more targeted approach to nonlinear problem resolution. Consequently, the dynamics of moving waves are frequently studied using expansion techniques. Equations (3) and (4) of the expansion method, whose methodology is presented in section 2.1, can be considered within this expansion method.

1. Step The ordinary differential equation's general solution is expressed as

$$u(\xi) = a_0 + \sum_{n=1}^{N} \left(a_n \left(\frac{G'}{G^2} \right)^n + b_n \left(\frac{G'}{G^2} \right)^{-n} \right), \tag{8}$$

$G = G(\xi)$ satisfies the prerequisites. If G meets the conditions, validity is guaranteed

$$\left(\frac{G'}{G^2} \right)' = \mu + \lambda \left(\frac{G'}{G^2} \right)^2, \tag{9}$$

where $\mu \neq 1$ and $\lambda \neq 0$ are integers, while $a_0, a_n, b_n, \quad (n = 1, 2, 3, 4.., N)$ are arbitrary constants. By balancing the nonlinear terms and the highest order derivative in the transformed ordinary differential equation, N is a positive integer.

2. Step By substituting equation (8) using equation (9) into equation (5), then collecting the coefficients of like powers of $\left(\frac{G'}{G^2}\right)^i$, $(i = 0, \pm1, \pm2, \pm3, \pm4, ...)$, by equating each coefficient to zero, a set of nonlinear algebraic equations is generated. To obtain the values of unknown constants a_0, a_n, b_n, $(n = 1, 2, 3, .., N)$, the resulting algebraic system is solved using Mathematical pakage.

3. Step There are three possibilities for the general solution of equation (9);
If $\mu\lambda > 0$,

$$\left(\frac{G'}{G^2}\right) = \sqrt{\frac{\mu}{\lambda}}\left(\frac{B\sin\left(\xi\sqrt{\mu}\lambda\right) + A\cos\left(\xi\sqrt{\mu}\lambda\right)}{B\cos\left(\xi\sqrt{\mu\lambda}\right) - A\sin\left(\xi\sqrt{\mu\lambda}\right)}\right). \tag{10}$$

If $\mu = 0, \quad \lambda \neq 0$,

$$\left(\frac{G'}{G^2}\right) = -\frac{C}{\lambda\left(C\xi + D\right)}. \tag{11}$$

If $\mu\lambda < 0$,

$$\left(\frac{G'}{G^2}\right) = -\frac{\sqrt{|\lambda\mu|}}{\lambda}\left(\frac{B + A\sinh\left(2\sqrt{|\mu\lambda|}\xi\right) + A\cosh\left(2\sqrt{|\mu\lambda|}\xi\right)}{-B + A\cosh\left(2\sqrt{|\mu\lambda|}\xi\right) + A\sinh\left(2\sqrt{|\mu\lambda|}\xi\right)}\right), \tag{12}$$

A and B are constants that are nonzero. By replacing the values of unknowns a_0, a_n, b_n, $(n = 1, 2, 3, 4, .., N)$ as well as the ratios equations (10), (11) and (12) into equation (8), three different types of solutions to the proposed equation can be found.

Exact solutions of the BLMP using $(1/G')$-expansion method

The solution given by equation (2) will be derived in this section using the $(1/G')$-expansion method. For equation (2), we can use the wave transformation as follows:

$$u(t, z, y, x) = u(\xi), \quad \xi = z + y + x - ct \tag{13}$$

equation (2) is changed to ordinary differential equation as follows under wave transformation:

$$-cu'' + u'''' - 6u'u'' = 0, \tag{14}$$

Integrating the aforementioned equation with regard to ξ yields this conclusion. Determining the integral using ξ as the variable is the procedure.

$$w - cu' + u''' - 3\left(u'\right)^2 = 0, \tag{15}$$

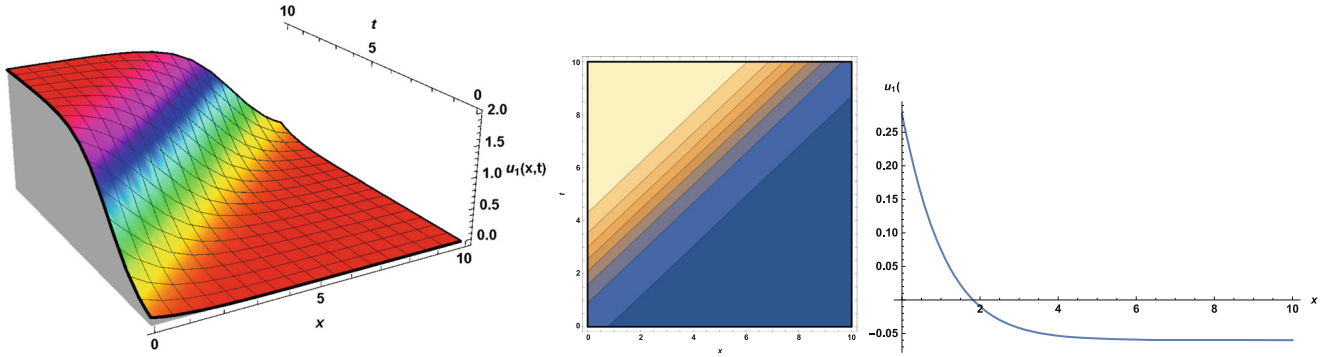


Fig. 1 2D, 3D and contour graph of equation (19) are shown for the parameter values $\mu = -1$, $\psi = 0.5$, $a_0 = 2$, and $\lambda = 1.03$

Upon applying the balancing principle to the analysis of equation (15), a particular connection is established. To ascertain the proper balance inside the equation, this concept is utilized, it is found that $N = 2$, therefore, equation (7) becomes:

$$u(\xi) = a_4\left(G'\right)^2 + a_3\left(G'\right) + a_2\left(\frac{1}{G'}\right)^2 + a_1\left(\frac{1}{G'}\right) + a_0, \tag{16}$$

equation (15) generates a polynomial in powers of $(\frac{1}{G'})$ when equation (16) and its derivatives are inserted. We obtain an algebraic system of equations by combining the coefficients of $(\frac{1}{G'})$ with exactly the same index. The equations are then established by equating each set of coefficients to zero. This procedure streamlines the solution and guarantees consistency.

$$\begin{aligned}
&\left(\tfrac{1}{G'}\right)^0: \quad w + c\mu a_3 - \lambda^2\mu a_3 + 6\lambda^2 a_1 a_3 - 3\mu^2 a_3^2 - 6\lambda\mu^2 a_4 + 24\lambda\mu a_1 a_4 + 24\lambda^2 a_2 a_4 = 0,\\
&\left(\tfrac{1}{G'}\right)^1: \quad -c\lambda a_1 + \lambda^3 a_1 + 12\lambda\mu a_1 a_3 + 12\lambda^2 a_2 a_3 + 12\mu^2 a_1 a_4 + 48\lambda\mu a_2 a_4 = 0,\\
&\left(\tfrac{1}{G'}\right)^2: \quad -c\mu a_1 + 7\lambda^2\mu a_1 - 3\lambda^2 a_1^2 - 2c\lambda a_2 + 8\lambda^3 a_2 + 6\mu^2 a_1 a_3 + 24\lambda\mu a_2 a_3 + 24\mu^2 a_2 a_4 = 0,\\
&\left(\tfrac{1}{G'}\right)^3: \quad 12\lambda\mu^2 a_1 - 6\lambda\mu a_1^2 - 2c\mu a_2 + 38\lambda^2\mu a_2 - 12\lambda^2 a_1 a_2 + 12\mu^2 a_2 a_3 = 0,\\
&\left(\tfrac{1}{G'}\right)^4: \quad 6\mu^3 a_1 - 3\mu^2 a_1^2 + 54\lambda\mu^2 a_2 - 24\lambda\mu a_1 a_2 - 12\lambda^2 a_2^2 = 0,\\
&\left(\tfrac{1}{G'}\right)^5: \quad 24\mu^3 a_2 - 12\mu^2 a_1 a_2 - 24\lambda\mu a_2^2 = 0,\\
&\left(\tfrac{1}{G'}\right)^6: \quad -12\mu^2 a_2^2 = 0,\\
&\left(G'\right)^1: \quad c\lambda a_3 - \lambda^3 a_3 - 6\lambda\mu a_3^2 + 2c\mu a_4 - 14\lambda^2\mu a_4 + 12\lambda^2 a_1 a_4 - 12\mu^2 a_3 a_4 = 0,\\
&\left(G'\right)^2: \quad -3\lambda^2 a_3^2 + 2c\lambda a_4 - 8\lambda^3 a_4 - 24\lambda\mu a_3 a_4 - 12\mu^2 a_4^2 = 0,\\
&\left(G'\right)^3: \quad -12\lambda^2 a_3 a_4 - 24\lambda\mu a_4^2 = 0,\\
&\left(G'\right)^4: \quad -12\lambda^2 a_4^2 = 0.
\end{aligned} \tag{17}$$

The following solution is constructed by using the above system of equations.

$$a_1 = 2\mu, \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad c = \lambda^2, \quad w = 0. \tag{18}$$

A new travelling wave solution to equation (2) is found by putting equation (18) and equation (13) into equation (16). The following is the outcome:

$$u_1(t, x, y, z) = \frac{2\mu}{-\frac{\mu}{\lambda} + \psi\cosh\left[\lambda\left(z + y + x - t\lambda^2\right)\right] - \psi\sinh\left[\lambda\left(x + y + z - t\lambda^2\right)\right]} + a_0, \tag{19}$$

or

$$u_1(t,x,y,z) = 2\lambda\left(\frac{\lambda\psi}{-\mathrm{e}^{\lambda(z+y+x-t\lambda^2)}\mu+\lambda\psi}-1\right)+a_0.$$

Precise soliton solutions of the BLMP using $\left(\frac{G'}{G^2}\right)$-expansion technique

With the help of the BLMP model, the proposed method is shown in detail.

Upon applying the balancing principle to the analysis of equation (15), a particular connection is established. To ascertain the proper balance inside the equation, this concept is utilized, it is found that $N = 2$, therefore, equation (8) becomes:

$$u(\xi) = b_2\left(\frac{G'}{G^2}\right)^{-2}+b_1\left(\frac{G'}{G^2}\right)^{-1}+a_2\left(\frac{G'}{G^2}\right)^{2}+a_1\left(\frac{G'}{G^2}\right)+a_0. \tag{20}$$

Equation (15) generates a polynomial in powers of $(\frac{G'}{G^2})$ when equation (21) and its derivatives are inserted. We obtain an algebraic system of equations by combining the coefficients of $(\frac{G'}{G^2})$ with exactly the same index. The equations are then established by equating each set of coefficients to zero. This procedure streamlines the solution and guarantees consistency.

$$\begin{aligned}
\left(\tfrac{G'}{G^2}\right)^{0}&: \quad w - c\mu a_1 + 2\lambda\mu^2 a_1 - 3\mu^2 a_1^2 + c\lambda a_2 - 2\lambda^2\mu a_2 + 12\lambda\mu a_1 a_2 - 3\lambda^2 a_2^2 + 48\lambda\mu b_1 b_2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{1}&: \quad -2c\mu b_1 + 16\lambda\mu^2 b_1 - 12\mu^2 a_1 b_1 + 24\lambda\mu a_2 b_1 + 12\lambda^2 a_1 b_2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{2}&: \quad -c\lambda a_1 + 8\lambda^2\mu a_1 - 6\lambda\mu a_1^2 + 6\lambda^2 a_1 a_2 - 12\mu^2 b_1^2 + 24\lambda^2 b_1 b_2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{3}&: \quad -2c\lambda b_1 + 40\lambda^2\mu b_1 - 24\lambda\mu a_1 b_1 + 12\lambda^2 a_2 b_1 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{4}&: \quad 6\lambda^3 a_1 - 3\lambda^2 a_1^2 - 24\lambda\mu b_1^2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{5}&: \quad 24\lambda^3 b_1 - 12\lambda^2 a_1 b_1 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{6}&: \quad -12\lambda^2 b_1^2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{-1}&: \quad 12\mu^2 a_2 b_1 + 2c\lambda b_2 - 16\lambda^2\mu b_2 + 24\lambda\mu a_1 b_2 - 12\lambda^2 a_2 b_2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{-2}&: \quad c\mu a_2 - 8\lambda\mu^2 a_2 + 6\mu^2 a_1 a_2 - 6\lambda\mu a_2^2 + 24\mu^2 b_1 b_2 - 12\lambda^2 b_2^2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{-3}&: \quad 2c\mu b_2 - 40\lambda\mu^2 b_2 + 12\mu^2 a_1 b_2 - 24\lambda\mu a_2 b_2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{-4}&: \quad -6\mu^3 a_2 - 3\mu^2 a_2^2 - 24\lambda\mu b_2^2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{-5}&: \quad -24\mu^3 b_2 - 12\mu^2 a_2 b_2 = 0,\\
\left(\tfrac{G'}{G^2}\right)^{-6}&: \quad -12\mu^2 b_2^2 = 0.
\end{aligned}$$

We get the following result after solving the above resulting system:

$$a_1 = 2\lambda,\ b_1 = 0,\ a_2 = 0,\ b_2 = 0,\ c = -4\lambda\mu,\ w = 0. \tag{21}$$

Case 1 If $\mu\lambda > 0$,the trigonometric solution to equation (2) is found by putting equation (22) and equation (13) into equation (21). The following is the outcome:

$$lu_2(t,x,y,z) = \frac{2\lambda\sqrt{\frac{\mu}{\lambda}}\left(A\cos\left(\sqrt{\lambda\mu}(4\lambda\mu t+z+y+x)\right)+B\sin\left(\sqrt{\lambda\mu}(4\lambda\mu t+z+y+x)\right)\right)}{-A\sin\left(\sqrt{\lambda\mu}(4\lambda\mu t+z+y+x)\right)+B\cos\left(\sqrt{\lambda\mu}(4\lambda\mu t+z+y+x)\right)}+a_0, \tag{22}$$

or

$$u_2(x,y,z,t) = a_0 + \frac{2\lambda\sqrt{\frac{\mu}{\lambda}}\left((B+iA)e^{2i\sqrt{\lambda\mu}(4\lambda\mu t+x+y+z)}+iA-B\right)}{-(A-iB)e^{2i\sqrt{\lambda\mu}(4\lambda\mu t+x+y+z)}+A+iB}.$$

Case 2 If $\mu = 0, \quad \lambda \neq 0,$

We get the following result after solving the above resulting system:

$$\mu = 0,\ a_1 = 2\lambda,\ b_1 = 0,\ a_2 = 0,\ b_2 = 0,\ c = -4\lambda\mu,\ w = 0. \tag{23}$$

The rational solution to equation (2) is found by putting equation (25) and equation (13) into equation (21). The following is the outcome:

$$u_3(t,z,y,x) = a_0 - \frac{2C}{D+C(z+y+x)}, \tag{24}$$

Case 3 If $\mu\lambda < 0$,the hyperbolic solution to equation (2) is found by substituting equation (22) and equation (13) into equation (21), below is the outcome:

$$u_4(x,y,z,t) = a_0 - \frac{2\sqrt{-\lambda\mu}\left(A\sinh\left(2\sqrt{-\lambda\mu}(4\lambda\mu t+x+y+z)\right)+A\cosh\left(2\sqrt{-\lambda\mu}(4\lambda\mu t+x+y+z)\right)+B\right)}{A\sinh\left(2\sqrt{-\lambda\mu}(4\lambda\mu t+x+y+z)\right)+A\cosh\left(2\sqrt{-\lambda\mu}(4\lambda\mu t+x+y+z)\right)-B}, \tag{25}$$

or

$$u_4(x,y,z,t) = 2\sqrt{-\lambda\mu}\left(\frac{2B}{B-Ae^{2\sqrt{-\lambda\mu}(4\lambda\mu t+x+y+z)}}-1\right)+a_0. \tag{26}$$

Stability Analysis of BLMP Equation

The stability of the provided equation (2) will be evaluated in this section using the linear stability test approach. This method entails analyzing the effects of minor perturbations on the solution. We can ascertain how the system responds to minor modifications by taking into account the model's perturbed solution. Analysis of the reaction to disturbances and how it affects stability will be the main emphasis. The analysis's findings will aid in

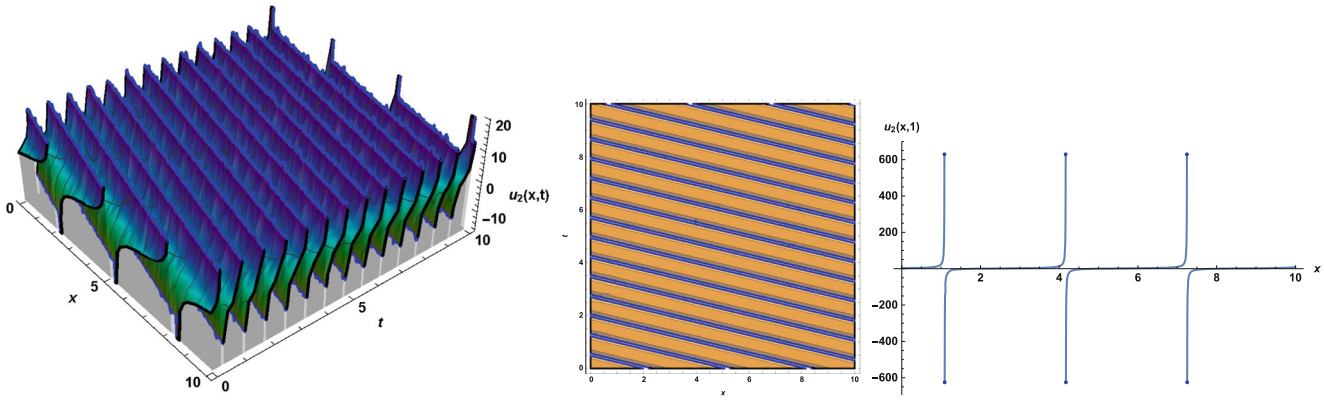


Fig. 2 3D, contour and 2D plot of equation (23) are shown for the parameter values $a_1 = 2,\ A = 0.2,\ B = 0.4,\ \lambda = 1.3,\ c = 1$and $\mu = 0.8$

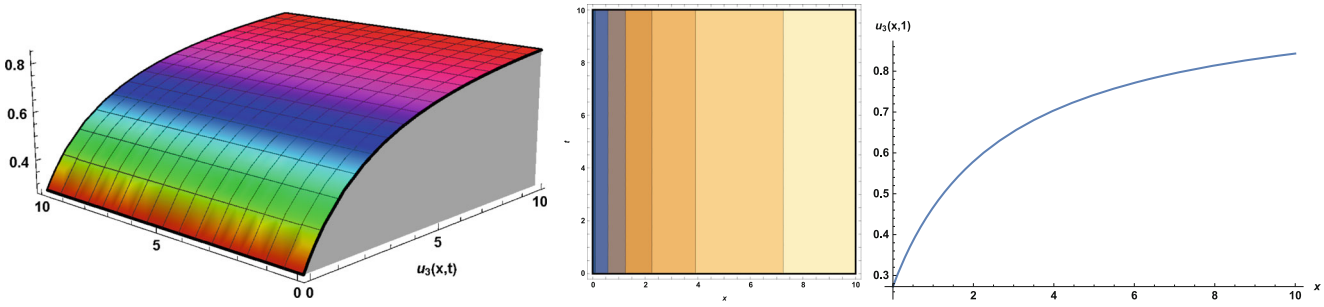


Fig. 3 3D, contour and 2D graph of equation (26) are shown for the parameter values $a_0 = 1,\ A = 8$ and $B = 6$

comprehending the equation's stability. In the end, this approach offers insightful information on the dynamic behavior of the model under perturbations.

$$u(t, z, y, x) = a + bv(t, z, y, x), \tag{27}$$

The solution of the BLMP equation (2) is clearly a stable state for any constant a. The following outcome is obtained when equation (29) is substituted into equation (2). We can confirm that the solution is stable according to this substitution. The procedure demonstrates that a is compatible with the BLMP model. The stability criteria for the provided solution is therefore confirmed by the resultant expression.

$$bv_{yt} + bv_{zt} + bv_{xxxy} + bv_{xxxz} - 3b^2v_xv_{xy} - 3b^2v_xv_{xz} - 3b^2v_{xx}v_y - 3b^2v_{xx}v_z = 0, \tag{28}$$

linearizing the equation above in b

$$v_{yt} + v_{zt} + v_{xxxy} + v_{xxxz} = 0, \tag{29}$$

Assume that the preceding equation permits a solution of the following form. Let's assume that the following is how the answer is expressed.

$$v(t, z, y, x) = fe^{i(cx+dy+ez-\mu t)}, \tag{30}$$

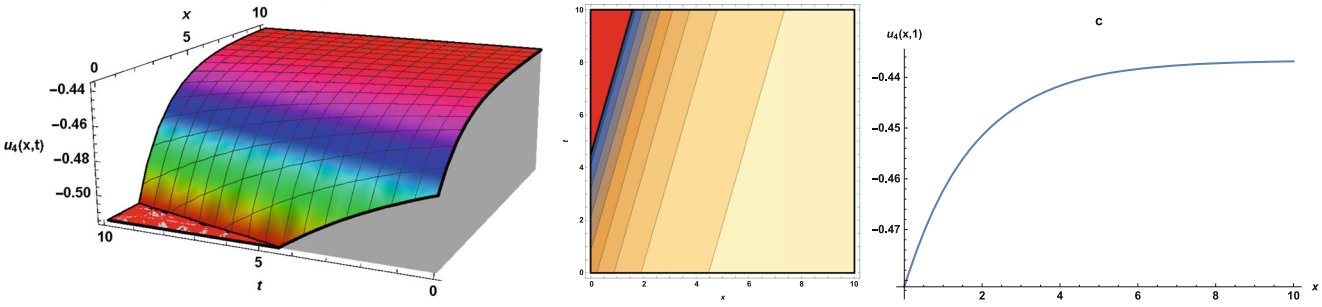


Fig. 4 3D, contour and 2D graph of equation (27) are shown for the parameter values $\mu = -0.009$, $a_0 = 0.1$, $\lambda = 8$, $A = 0.1$ and $B = 0.01$

The normalized wave numbers are c, d and e in this instance. The resultant expression may be obtained by substituting equation (32) into equation (31). In this stage, the original equation is updated to include the normalized wave numbers for additional analysis.

$$\mu(c, d, e) = c^3, \tag{31}$$

To comprehend their behavior, the transmission relationships given in equation (33) are examined. Whether the solution amplifies or reduces with time depends critically on the sign of $\mu(c, d, e)$. Self-phase modulation, group velocity dispersion, and stimulated Raman scattering are some of the variables that affect the stability of the steady-state solution for the dispersion relation in equation (33). Even with slight disturbances, the wave number c maintains its actual values when when $\mu(c, d, e) \neq 0$. But when $\mu(c, d, e) = 0$, the wave numbers become closer to infinity, which makes the steady-state solution unstable and causes the distortion to increase exponentially. The crucial moment at which the system changes from stability to instability is shown by this divergence.

Thus, the variation of $\mu(c, d, e)$ can be directly related to stability in these solutions. Modulation stability is readily apparent for $c \neq 0$, guaranteeing the solution's resilience. The connection between the parameters demonstrates how little adjustments may have a big impact on how the system behaves as a whole. In the end, our research shows how crucial these dynamics are to preserving stability.

Result and discussion

BLMP model serves as a generalization of the shallow water wave model to higher-dimensional spaces. It is used to explain the incompressible properties of the ion acoustic wave and the stratified oceanic interior wave. New traveling wave solutions of this equation have been successfully obtained using the $\left(\frac{G'}{G^2}\right)$-expansion method and $\left(\frac{1}{G'}\right)$-expansion method. The traveling wave solutions obtained in this study differ from the results reported in the existing literature for the BLMP equation in terms of form and parameters [44–53]. **With the selection of appropriate fixed parameters, two and three dimensional visualizations and contour graph representations of all solutions were successfully created. Figs 1, 2, 3 and 4, presented in relation to the wave solution, have been generated for solutions exhibiting soliton solution characteristics, taking into account special cases where the parameters are stable. These graphs are presented for the purpose of detailed examination of the solution behavior and visual analysis of the results. These analytical solutions demonstrate an effective method for solving nonlinear differential equations and offer potential benefits, particularly in engineering applications such as fluid dynamics and plasma explosions.**

The modified $\left(\frac{1}{G'}\right)$-expansion technique and $\left(\frac{G'}{G^2}\right)$-expansion technique used in this study were applied to obtain analytical solutions for the (3+1)-dimensional BLMP equation. While both methods enable the gener-

ation of traveling wave solutions, they exhibit distinct differences in terms of structure and functionality. The modified $\left(\frac{1}{G'}\right)$-expansion method is more practical in terms of application because it involves relatively simpler transformations. The solutions obtained with this method are generally hyperbolic in form and are physically meaningful and directly interpretable. However, this technique is mostly valid for producing solutions within limited parameter ranges and has some limitations in terms of generalizability. On the other hand, the $\left(\frac{G'}{G^2}\right)$-expansion method makes it possible to obtain more complex solution structures. The solutions obtained using this method are generally in the form of hyperbolic, trigonometric, or rational functions, which are physically meaningful and directly interpretable. Although the transformation steps require more technical knowledge and attention, this method can be applied to a wider parameter range and offers flexibility in modeling the different behaviors of physical systems.

Travelling wave solutions in applied sciences are an important tool for modelling and analysing the behaviour of incompressible fluids. In addition to simulating a system's time-varying behavior, these waves can also be utilized to depict stationary or non-stationary wave systems. By concentrating on one of the various travelling wave solutions generated by the two methods in this study, fluid dynamics can be discussed. equation (19) can be taken into account for discussion. The propagation of a wave or waves in a particular medium can be described using traveling wave solutions, along with the fluid's velocity distribution. In addition to examining the wave behavior in relation to the fluid's characteristics, it also enables the investigation of how the fluid's velocity varies at a specific time and location. Velocity, one of the inherent characteristics of incompressible fluids, can demonstrate how a traveling wave alters through time and space.

In particular, the transformation provided by equation (13) is utilized to explain wave-like behaviour while approaching the solution of various issues involving waves and vibrations. The term ct denotes a signal that varies over time, whereas the terms x, y, and z denote a signal that travels over space as a wave. The wave velocity is physically represented by the parameter c in a typical wave transform. When equation (18) is considered, it is shown as $c = \lambda^2$. Additionally, it can be inferred from an analysis of the traveling wave solution provided by equation (19) that the λ parameter directly influences the wave velocity even if it also affects the number of waves in the x, y, and z directions. To better understand this situation, all of the variables in equation (19) can be thought of as constants, with the exception of λ. For various values of λ, the simulation below can be examined to see how the wave solution evolves over time, to comprehend how the wave acts and propagates, and to learn more about the wave characteristic.

Equation (19), the model of the time-dependent behavior of incompressible fluids, can be used to examine the visual of the nonlinear wave distribution in the x direction in Figure 5. The nonlinear wave distribution in the y and z directions will, similarly, can be stated to have similar properties. In addition, the wave speed changes according to λ, with the value decreasing when it is $0 < \lambda < 1$ and increasing when it is $\lambda > 1$. Diffusion and absorption are intimately correlated with both an increase and a decrease in wave velocity. In addition, wave velocity is related to wave energy per unit time or space. Scientifically speaking, a wave with high energy moves and spreads faster. This body of knowledge suggests that equation (19) for $\lambda > 1$ can represent very energetic waves travelling. Similar discussions can be made in other generated traveling wave solutions. In this study, the analytical solution presented in equation (19) can be given physical meaning as the velocity component of an incompressible fluid in a specific direction. This solution is a traveling wave solution of the (3+1) dimensional BLMP equation and represents a time-dependent structural distortion propagating within the fluid. Furthermore, the dependence of the u_1 component on the $\boldsymbol{y}$ and $\boldsymbol{z}$ variables indicates that the solution is not merely an x-directed wave; it may represent a three-dimensional spatial structure with a vorticity-driven flow. This situation can be explained by the vorticity components derived from the solution, $\omega_2 = \frac{\partial u_1}{\partial z}$ and $\omega_3 = -\frac{\partial u_1}{\partial y}$. The non-zero vorticity structure indicates that the flow has a rotational character.

This study may be of interest to researchers working on nonlinear wave dynamics and to engineering and physics experts in applied fields such as fluid mechanics, optical communication, and plasma physics. The (3+1)-dimensional equation (2) examined possesses a robust structure that can be utilized in modeling incompressible fluids and high-energy wave systems. The analytical solutions obtained explain the role of physical parameters

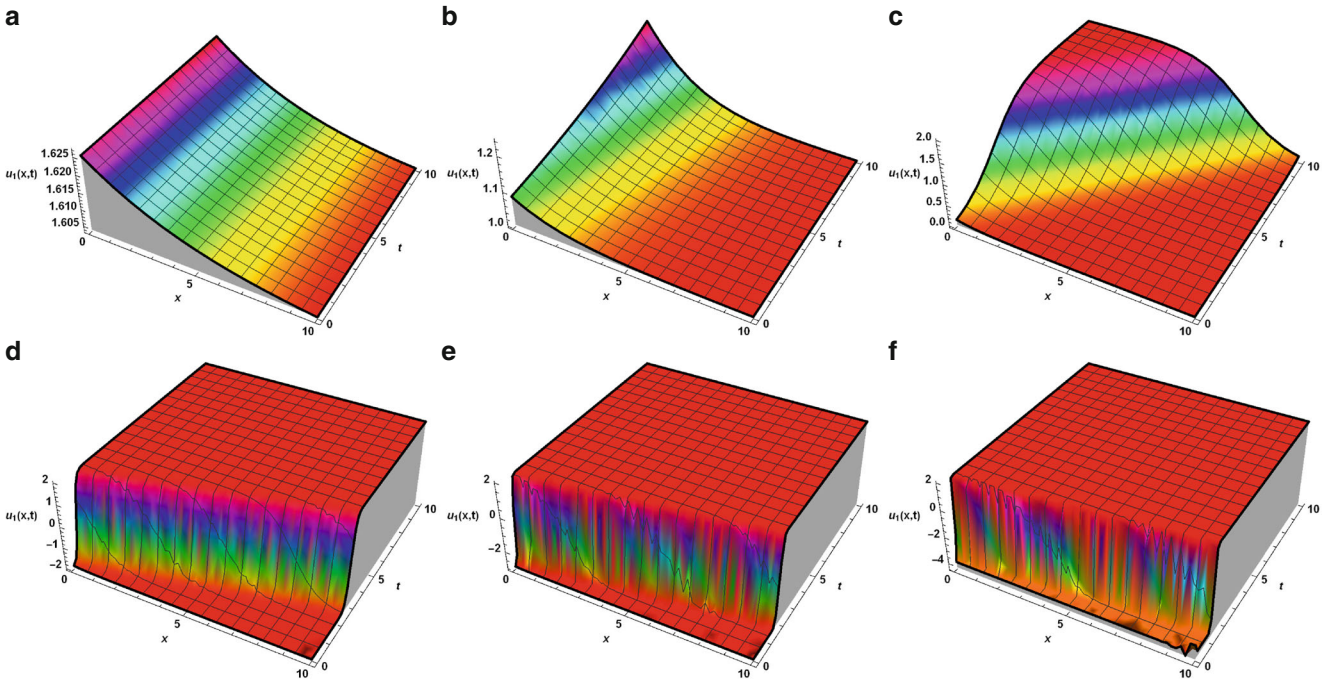


Fig. 5 3D plot of equation (19), **a** $\lambda = 0.2$, **b** $\lambda = 0.5$, **c** $\lambda = 1$, **d** $\lambda = 2$, **e** $\lambda = 2.5$, **f** $\lambda = 3$ for $\mu = -1, , \psi = 0.5, a_0 = 2, y = 1, z = 1$

such as soliton behavior, wave velocity–energy relationship, diffusion, and absorption effects on the system and are directly related to real-world applications such as optical pulse transmission, energy transfer, and wave control. The study is anticipated to contribute to both the theoretical characterization of these systems and engineering design processes.

Conclusion

In this study, the (3+1)-dimensional BLMP equation, which is a high-dimensional extension of the shallow water wave model, is considered to analyze the behavior of incompressible fluids by investigating the nonlinear dispersion mechanism with the influence of the wave speed factor. Two analytical methodologies are proposed, yielding a wide variety of moving wave solutions under certain restrictive conditions. The stability of the obtained solutions is evaluated using a linear stability analysis method that examines the effects of small-amplitude perturbations, and the necessary constants for the system to remain stable are appropriately determined. One of the original contributions of this study is the comparative analysis of the advantages and limitations of the two analytical methods used. In this context, a strong physical correlation between diffusion, absorption, and wave velocity has been revealed, and it has been concluded that high-energy waves propagate faster due to the relationship between wave velocity and energy per unit space or time. Additionally, the steady-state stability of the equation has been linked to important physical processes such as stimulated Raman scattering, group velocity dispersion, and intrinsic phase modulation. In conclusion, the analytical solutions and stability analyses obtained provide a solid foundation for modeling and simulating the dynamics of incompressible fluids, thereby contributing to the development of safe and effective products in engineering design. In the future, expanding the model with different physical effects and numerical methods will increase its application potential in this field.

Author Contribution **Asif Yokus:** Supervision, Investigation, Writing–review editing, Methodology, Software, Conceptualization, Project administration, Visualization, Formal analysis. **Muhammad Abubakar Isah:** Investigation, Software, Methodology, Conceptualization, Resources, Writing – review editing. **Munevver Tuz:** Writing – review and editing, Writing – original draft, Investigation.

Data Availability Statement The authors confirm that the data underpinning the findings of this study are included within the article, with the figures providing a clear representation of the results.

Declarations

Conflict of interest The authors declare that there are no financial, personal, or professional conflicts of interest that could have influenced the research, authorship, or publication of this article.

Ethical standards The authors confirm that this work adheres to established ethical standards.

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