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Research paper

Unveiling novel insights: Nonlinear dispersion dynamics of optical solitons through new φ^6-Model expansion in the Klein–Gordon equation

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ABSTRACT

This study employs a nonlinear differential equation to model diverse phenomena, encompassing dislocation movement in crystals, characteristics of elementary particles, and the propagation of fluxions in Josephson junctions, with the Klein–Gordon equation serving as an illustrative example. The φ^6-model expansion method, a multitude of solution types are explicitly obtained, which encompass kink-type solitons, recognized as topological solitons within the realm of water waves. Notably, these solitons exhibit velocities independent of wave amplitude, alongside other variations like dark, singular, periodic, and combined singular soliton solutions. The research outcomes hold the potential to enhance the nonlinear dynamical characteristics of the Klein–Gordon equation. The suggested φ^6-model expansion technique provides a pragmatic and efficient strategy for addressing a wide range of nonlinear partial differential equations. The findings are visually presented through insightful graphs that elucidate the dynamic aspects of the results, demonstrating the accuracy of the obtained solutions when applied to the Klein–Gordon equation. The physical properties of surface waves are comprehensively analyzed, with a particular focus on Rayleigh waves. By modeling the regular oscillations, energy transfer and complex behavior of Rayleigh waves under nonlinear effects, this study provides theoretical support that these waves can exhibit solitary wave properties under certain boundary conditions.

1. Introduction

The origins of partial differential equations (PDEs) trace back to their emergence in the study of surfaces within geometry [1–3] and the exploration of various mechanical problems. During the late 19th century, notable mathematicians worldwide became actively involved in delving into a broad spectrum of issues posed by PDEs. Beyond their historical context, PDEs, particularly nonlinear partial differential equations, have proven crucial in describing phenomena across physics, biology, chemistry, and other scientific domains. The nonlinear behavior inherent in many physical systems harbors intriguing features that demand specialized methods for exploration.

The primary motivation behind this study stems from the pervasive presence of PDEs in mathematical analyses, addressing challenges in science and engineering while elucidating fundamental natural principles [4]. As we delve into this research, it becomes increasingly apparent that the investigation of nonlinear evolution equations (NLEEs) and PDEs demands precision and a diverse set

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of methodologies. Among these techniques, the inverse scattering transform approach [5] stands out as an effective means to seek precise solutions for both NLEEs and PDEs, underscoring the rich landscape of mathematical tools available for this exploration. others include; the Homoclinic method [6,7], the sinh-Gordon function technique [8,9], the generalized exponential rational function method [10], the general projective Riccati method [11], the auxiliary equation approach [12], An alternative technique [13], the symbolic computation [14], F -expansion method [15,16], the bilinear neural network method [17,18], the Bernoulli sub-equation function technique [19], the sub-equation analytical technique [20], the modified sub-equation technique [21], the unified auxiliary equation method [22], the auto-Bäcklund transformation method [23], unified solver method [24], residual power series technique [25], modified Laplace decomposition method [26], the newly developed φ^6–model expansion technique [27–29] and so on.

This investigation is centered on the intricate realm of the nonlinear Klein–Gordon equation, a subject that has captivated the interest of researchers and scholars alike. The equation's significance extends to the realms of studying solitons and delving into the fascinating domain of condensed matter physics [30,31]. Furthermore, it has played a pivotal role in unraveling the complexities surrounding soliton interactions within collisionless plasmas, investigating the reappearance of initial states, and delving into the intricate structure of nonlinear wave equations [32]. As we embark on this exploration, we aim to illuminate the multifaceted nature of the Klein–Gordon equation and contribute to the ongoing discourse in these captivating scientific domains. This article focuses on the analytical solution of the Klein–Gordon equation (KGE) [33];

$$u_{tt} - \Delta u + au = f(u), \quad 0 < t \leq T, \ x \in T^d \subset R^d, \tag{1}$$

in which $u = u(t, x)$ denotes the wave motion at time t and location x, $d = 1, 2, 3$ indicates the dimension of x, $0 < a$ is a provided quantity which can be described physically as the particle's mass, while $f(u) : R \to R$ is an arbitrary nonlinear function. The NKGE is universally well-posed for initiating data $\left(u_0(x), V_0(x)\right) \in H^{\alpha}(T^d) \times H^{\alpha-1}(T^d)$ for $1 \leq \alpha$ in higher dimensions. There are several well-known instances resulting from various selections of $f(u)$. For the highly rare situation $f(u) = 0$, the aforementioned Klein–Gordon model is regarded as the relativistic version of the Schrödinger equation, because it accurately represents spinless relativistic composite elements like the pion and the Higgs boson [33]. If $f(u) = au^n$ is chosen, the equation is known as the Klein–Fock–Gordon equation, and is known as the quadratic nonlinearity for $n = 2$ and the cubic nonlinearity for $n = 2$ [34]. In the situation where $f(u) = bu^3$ with a dimensionless parameter b (negative and positive for focusing self-interaction and defocusing, respectively). If $f(u) = sin(u)$, Eq. (1) correlates to the sine-Gordon equation, which emerges in numerous physical applications [35], the equation includes the cubic wave model depending on the choice of $a = 0$, and this system is commonly used in plasma physics [36]. The case $f(u) = e^u$ is known by the Liouville equation, a fundamental tool in classical mechanics for understanding the evolution of systems [37]. The Boussinesq equation that describes the propagation of long waves with tiny amplitudes can be analyzed as $f(u) = (u^2)_{xx} + u_{xxxx}$ [38,39].

In this article, our focal point is the exploration of cutting-edge research pertaining to the φ^6–model expansion method. Specifically, we aim to investigate precise solutions for the well-known $(1 + 1)-$dimensional Klein–Gordon (KG) equation by taking $f(u) = bu^3$ with a dimensionless parameter b (negative and positive for focusing self-interaction and defocusing, respectively) as shown in Eq. (11). The overarching goal is to meticulously assess and underscore the advantages and effectiveness of the φ^6–model expansion method in handling this complex equation. The KG equation, renowned for its versatility, finds applications in modeling a spectrum of nonlinear phenomena, ranging from the intricate propagation of dislocations in crystals to the nuanced behavior of elementary particles and the fascinating dynamics of fluxions in Josephson junctions. Through this endeavor, we aspire not only to contribute to the evolving body of knowledge but also to showcase the practical utility of the φ^6-model expansion method in unraveling the complexities inherent in real-world physical phenomena. In this study, the Klein–Gordon equation is examined utilizing the recently developed φ^6-model expansion technique, which leads to the recovery of optical solitary wave solutions. he φ^6-model expansion method offers an efficient, flexible and physically meaning-rich strategy for modeling nonlinear dynamics. The φ^6–model expansion method is chosen for this study due to its variety of solutions, ease of implementation and suitability for physical systems. This approach offers a fresh perspective on understanding the dynamics of optical solitons within the context of the Klein–Gordon equation. The Klein–Gordon equation can shed light on the energy transport mechanism in nonlinear wave dynamics. In this study, the Klein–Gordon equation will be used as a tool to explain the velocity and energy density of wave motion. In particular, solitary waves and soliton solutions of the equation will be used to explain the nonlinear effects of Rayleigh waves related to elastic material dynamics. It is known that Rayleigh waves can exhibit soliton-like behavior under certain conditions [40]. The Klein–Gordon equation provides a theoretical basis for the formation of solitons. Therefore, the Klein–Gordon equation can be used to study the similar dynamics of Rayleigh waves.

In charting the course for this endeavor, a strategic blueprint unfolds as follows. Section 2 lays the foundation with a comprehensive exploration of the φ^6-model expansion method, providing the groundwork for subsequent developments. The narrative intensifies in Section 3, where we skillfully wield the φ^6-method to fashion an innovative rendition of the Klein–Gordon equation, unveiling a tapestry of fresh and dynamic traveling wave solutions. To enrich our understanding, Section 3 introduces a visual symphony of 3D, 2D, and contour graphs, painting a vivid portrait of the physical structure inherent in these traveling wave solutions. In Section 4, the spotlight turns to the stability analysis of the soliton solutions, unraveling their nuanced behaviors. As our research culminates in Section 5, we draw astute conclusions that encapsulate the essence of our discoveries, beckoning further exploration into the captivating realm of nonlinear wave phenomena.

2. The φ^6-model expansion method

As outlined in the Refs. [41–46], the procedural steps for implementing the φ^6-model expansion technique encompass:

Step-1: Assuming the nonlinear partial differential equation describing $u = (x, t)$ takes the following form:

$$Q(u, u_x, u_t, u_{xt}, u_{xx}, \ldots) = 0. \tag{2}$$

Here, Q is expressed as a polynomial function of $u(x, t)$, involving both the highest order partial derivatives and non-linear terms inherent in its formulation.

Step-2: Through the utilization of the wave transformation

$$u(x, t) = u(\mu), \quad \mu = x - wt. \tag{3}$$

In the scenario where w is indicative of the wave velocity, Eq. (2) can be converted into below nonlinear ordinary differential equation.

$$\Omega(u, u', uu', u'', \ldots) = 0, \tag{4}$$

here the prime notation is used to indicate derivatives with respect to μ.

Step-3: Assuming the existence of a formal solution to Eq. (4):

$$u(\mu) = \sum_{j=0}^{2C} \alpha_i \varphi^j(\mu), \tag{5}$$

C can be determined through the application of the balancing rule, where α_j (for $j = 0, 1, 2, \ldots, C$) are constants to be established, and $\varphi(\mu)$ adheres to the φ^6-model expansion in the following format:

$$\begin{aligned} \varphi'^2(\mu) &= h_0 + h_2\varphi^2(\mu) + h_4\varphi^4(\mu) + h_6\varphi^6(\mu), \\ \varphi''(\mu) &= h_2\varphi(\mu) + 2h_4\varphi^3(\mu) + 3h_6\varphi^5(\mu). \end{aligned} \tag{6}$$

In this context, the constants h_j (where $j = 0, 2, 4, 6$) are initially arbitrary and will be determined in subsequent stages.

Step-4: It is established that the solution to Eq. (6) takes the form:

$$\varphi(\mu) = \frac{Q(\mu)}{\sqrt{fQ^2(\mu) + g}}. \tag{7}$$

$Q(\mu)$ denotes the solution to the Jacobi elliptic equation, ensuring the validity of $0 < fQ^2(\mu) + g$

$$Q'^2(\mu) = l_0 + l_2Q^2(\mu) + l_4Q^4(\mu). \tag{8}$$

Here, the constants l_j (for $j = 0, 2, 4$) are unspecified and will be established through calculation. The values of g and f are provided by:

$$\begin{aligned} f &= \frac{h_4(l_2 - h_2)}{(l_2 - h_2)^2 + 3l_0l_4 - 2l_2(l_2 - h_2)}, \\ g &= \frac{3l_0h_4}{(l_2 - h_2)^2 + 3l_0l_4 - 2l_2(l_2 - h_2)}. \end{aligned} \tag{9}$$

Subject to the specified condition

$$h_4^2(l_2 - h_2)[9l_0l_4 - (l_2 - h_2)(2l_2 + h_2)] + 3h_6[-l_2^2 + h_2^2 + 3l_0l_4]^2 = 0. \tag{10}$$

Step-5: The computation of Jacobi elliptic solutions for Eq. (8) is feasible under the condition $0 < N < 1$. Upon inserting Eq. (7) and (8) into Eq. (5), precise solutions for Eq. (2) can be derived.

Function	$N \to 1$	$N \to 0$	Function	$N \to 1$	$N \to 0$
$sn(\mu, N)$	$tanh(\mu)$	$sin(\mu)$	$ds(\mu, N)$	$csch(\mu)$	$csc(\mu)$
$cn(\mu, N)$	$sech(\mu)$	$cos(\mu)$	$sc(\mu, N)$	$sinh(\mu)$	$tan(\mu)$
$dn(\mu, N)$	$sech(\mu)$	1	$sd(\mu, N)$	$sinh(\mu)$	$sin(\mu)$
$ns(\mu, N)$	$coth(\mu)$	$csc(\mu)$	$nc(\mu, N)$	$cosh(\mu)$	$sec(\mu)$
$cs(\mu, N)$	$csch(\mu)$	$cot(\mu)$	$cd(\mu, N)$	1	$cos(\mu)$

3. Utilization of the φ^6 method in the analysis of the Klein– Gordon equation

The aim of this segment is to explore recent research associated with the φ^6-model expansion method, specifically in investigating precise solutions for the (1+1)-dimensional cubic Klein–Gordon (cKG) equation. The goal is to assess the method's benefits and

effectiveness. The cKG equation is employed to simulate various nonlinear phenomena, such as the movement of dislocations in crystals, the characteristics of elementary particles, and the propagation of fluxions in Josephson junctions.

$$u_{tt} - c^2 u_{xx} + au + bu^3 = 0. \tag{11}$$

In the scenario where the amplitude of the relevant wave mode is denoted by u and $u = u(x,t)$, with a and b representing constants, Eq. (11) becomes the Phi-4 equation [47], which has played a significant role in nuclear and particle physics throughout the years. The cKG equation is used to describe the propagation of dislocations inside crystals, the Bloch wall motion of magnetic crystals, the propagation of a "splay wave" over a dielectric membrane, the unitary theory for elementary particles, and the transmission of magnetic flux on a Josephson line, among other topics. More information is provided in [48,49] and the sources cited there. Khan et al. [50] used the Modified simple equation method to get the traveling wave solutions to the $(2+1)$-dimensional cKG model. Traveling wave solutions are expressed using hyperbolic and trigonometric functions with arbitrary parameters. The homogeneous balancing approach, the Jacobi elliptic expansion method, and the auxiliary equation method are used to get the Jacobi doubly periodic wave solutions to the generalized Klein–Gordon equation by Hong-Cai et al. [51], other method include the use of modified Exp-function method [52], the tanh method [53], the solitary wave ansatz [54], and A numerical approach based on a three-time level finite-difference [55]. A comparison with the results in the above studies shows that almost all of our solutions are completely different from theirs.

Eq. (11) undergoes simplification into the subsequent ordinary differential equation through the application of the traveling wave transformation $u(x,t) = u(\mu) = u(x - wt)$

$$\left(w^2 - c^2\right) u'' + bu^3 + au = 0. \tag{12}$$

The value of $C = 1$ is derived through the equilibrium principle involving u'' and u^3. Consequently, the solution can be represented in the following form:

$$u(\mu) = \alpha_0 + \alpha_1 \varphi(\mu) + \alpha_2 \varphi^2(\mu). \tag{13}$$

Here, the constants α_0, α_1, and α_2 are undetermined and will be obtained through subsequent analysis.

Upon incorporating Eqs. (13) and (6) into Eq. (12) and equalizing the coefficients of all powers of $\varphi^j(\mu)$ to zero, where $j = 0,1,2,3,4,5,6$, the following set of algebraic equations is obtained:

$$\begin{aligned}
&\varphi^0(\mu) : a\alpha_0 + \alpha_0^3 b - 2\alpha_2 c^2 h_0 + 2\alpha_2 h_0 w^2 = 0,\\
&\varphi^1(\mu) : a\alpha_1 + 3\alpha_0^2\alpha_1 b + \alpha_1 c^2\left(-h_2\right) + \alpha_1 h_2 w^2 = 0,\\
&\varphi^2(\mu) : a\alpha_2 + 3\alpha_0\alpha_1^2 b + 3\alpha_0^2\alpha_2 b - 4\alpha_2 c^2 h_2 + 4\alpha_2 h_2 w^2 = 0,\\
&\varphi^3(\mu) : \alpha_1^3 b + 6\alpha_0\alpha_2\alpha_1 b - 2\alpha_1 c^2 h_4 + 2\alpha_1 h_4 w^2 = 0,\\
&\varphi^4(\mu) : 3\alpha_0\alpha_2^2 b + 3\alpha_1^2\alpha_2 b - 6\alpha_2 c^2 h_4 + 6\alpha_2 h_4 w^2 = 0,\\
&\varphi^5(\mu) : 3\alpha_1\alpha_2^2 b - 3\alpha_1 c^2 h_6 + 3\alpha_1 h_6 w^2 = 0,\\
&\varphi^6(\mu) : \alpha_2^3 b - 8\alpha_2 c^2 h_6 + 8\alpha_2 h_6 w^2 = 0.
\end{aligned} \tag{14}$$

The solution presented below is derived upon solving the aforementioned system.

$$\alpha_0 = 0, \quad \alpha_1 = \frac{\sqrt{2}\sqrt{c^2 h_4 - h_4 w^2}}{\sqrt{b}}, \quad \alpha_2 = 0, \quad h_6 = 0, \quad a = h_2\left(c^2 - w^2\right). \tag{15}$$

The precise solutions for Eq. (11) can be determined by utilizing Eqs. (7), (13), and (15), together with the Jacobi elliptic functions listed in the aforementioned table.

1. Assuming $l_0 = 1$, $l_2 = -(1+N^2)$, $l_4 = N^2$, where $0 < N < 1$, the function $Q(\mu) = sn(\mu, N)$ is obtained, leading to the following:

$$u_{1,0}(x,t) = \alpha_1\left(\frac{sn(\mu,N)}{\sqrt{f\,(sn(\mu,N))^2 + g}}\right), \tag{16}$$

if $l_0 = 1$, $l_2 = -(1+N^2)$, $l_4 = N^2$, $0 < N < 1$, then $Q(\mu) = cd(\mu, N)$ therefore

$$u_{1,1}(x,t) = \alpha_1\left(\frac{cd(\mu,N)}{\sqrt{f\,(cd(\mu,N))^2 + g}}\right). \tag{17}$$

In this scenario, when $\mu = x - wt$, and $sn(\mu,N)$ and $cd(\mu,N)$ represent the Jacobi elliptic functions with a solution modulus of $0 < N < 1$, these functions take on the form of either trigonometric or hyperbolic functions, as detailed in the aforementioned table. The expressions for f and g in Eq. (9) are given by:

$$f = \frac{h_4\left(h_2 + N^2 + 1\right)}{-h_2^2 + N^4 - N^2 + 1},$$
$$g = -\frac{3h_4}{-h_2^2 + N^4 - N^2 + 1},$$

under the restriction condition

$$h_4^2\left(-h_2-N^2-1\right)\left(-\left(-h_2+2N^2-1\right)\left(h_2+N^2-2\right)\right)=0.$$

By taking Eq. (16) into consideration, when $N\to 1$, then $sn(\mu,1)=tanh(\mu)$, the kink soliton is retrieved

$$u_{1,2}(x,t)=-\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\tanh(tw-x)}{\sqrt{b}\sqrt{\frac{h_4\left(3-(h_2+2)\tanh^2(tw-x)\right)}{h_2^2-1}}}, \tag{18}$$

where $\mu=x-wt$, such that

$$h_4^2\left(-h_2-2\right)\left(\left(h_2-1\right)^2\right)=0.$$

As N tends toward 0, $sn(\mu,0)=sin(\mu)$, resulting in the acquisition of the periodic solution.

$$u_{1,3}(x,t)=-\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\sin(tw-x)}{\sqrt{b}\sqrt{-\frac{h_4\left(h_2\sin^2(tw-x)+\sin^2(tw-x)-3\right)}{h_2^2-1}}}, \tag{19}$$

such that

$$h_4^2\left(-h_2-1\right)\left(-\left(-h_2-1\right)\left(h_2-2\right)\right)=0.$$

Similarly, examining the cases where $cd(\zeta,1)=1$ and $cd(\zeta,0)=\cos(\zeta)$ in Eq. (17), diverse solutions can be attained.

2. Assuming $l_0=1-N^2$, $l_2=2N^2-1$, $l_4=-N^2$, with $0<N<1$, we conclude that $Q(\mu)=cn(\mu,N)$, thus resulting in:

$$u_{2,1}(x,t)=\alpha_1\left(\frac{cn(\mu,N)}{\sqrt{f\left(cn(\mu,N)\right)^2+g}}\right), \tag{20}$$

where the values of f and g are ascertained by

$$f=\frac{h_4\left(h_2-2N^2+1\right)}{-h_2^2+N^4-N^2+1},$$
$$g=\frac{3h_4\left(N^2-1\right)}{-h_2^2+N^4-N^2+1},$$

under the specified constraint

$$h_4^2\left(-h_2+2N^2-1\right)\left(-\left(-h_2+2N^2-1\right)\left(h_2+2\left(2N^2-1\right)\right)-9\left(1-N^2\right)N^2\right)=0.$$

If $N\to 1$, $cn(\mu,1)=\mathrm{sech}(\mu)$ then the trivial solution is retrieved and the constraint condition is as follows.
provided that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)=0.$$

As N approaches 0, the solution becomes periodic.

$$u_{2,2}(x,t)=\frac{\sqrt{2}\sqrt{(c-w)(c+w)}\cos(tw-x)}{\sqrt{b}\sqrt{-\frac{\left(h_2\cos^2(tw-x)+\cos^2(tw-x)-3\right)}{h_2^2-1}}}, \tag{21}$$

such that

$$h_4^2\left(-h_2-1\right)\left(-\left(-h_2-1\right)\left(h_2-2\right)\right)=0.$$

3. If $l_0=N^2-1$, $l_2=2-N^2$, $l_4=-1$, $0<N<1$, then $Q(\mu)=dn(\mu,N)$ which gives

$$u_{3,1}(x,t)=\alpha_1\left(\frac{dn(\mu,N)}{\sqrt{f\left(dn(\mu,N)\right)^2+g}}\right), \tag{22}$$

where the values of f and g are ascertained by

$$f=\frac{h_4\left(h_2+N^2-2\right)}{-h_2^2+N^4-N^2+1},$$
$$g=-\frac{3h_4\left(N^2-1\right)}{-h_2^2+N^4-N^2+1},$$

under the restriction condition

$$h_4^2\left(-h_2-N^2+2\right)\left(-\left(-h_2-N^2+2\right)\left(h_2+2\left(2-N^2\right)\right)-9\left(N^2-1\right)\right)=0.$$

If $N\to 1$, then the rational solution is obtained

$$u_{3,2}(x,t)=\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}}{\sqrt{b}\sqrt{-\frac{h_4}{h_2+1}}}, \tag{23}$$

provided that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)=0.$$

If $N\to 0$, then the rational solution is obtained

$$u_{3,3}(x,t)=\frac{\sqrt{2}\sqrt{c^2h_4-h_4w^2}}{\sqrt{b}\sqrt{-\frac{(h_2-2)h_4}{h_2^2-1}-\frac{3h_4}{h_2^2-1}}}, \tag{24}$$

such that

$$h_4^2\left(2-h_2\right)\left(9-\left(2-h_2\right)\left(h_2+4\right)\right)=0.$$

4. If $l_0=N^2$, $l_2=-\left(1+N^2\right)$, $l_4=1$, $0<N<1$, then $Q(\mu)=ns(\mu,N)$ or $Q(\mu)=dc(\mu,N)$ then

$$u_{4,0}(x,t)=\alpha_1\left(\frac{ns(\mu,N)}{\sqrt{f\left(ns(\mu,N)\right)^2+g}}\right), \tag{25}$$

or

$$u_{4,1}(x,t)=\alpha_1\left(\frac{dc(\mu,N)}{\sqrt{f\left(dc(\mu,N)\right)^2+g}}\right), \tag{26}$$

where the values of f and g are ascertained by

$$f=\frac{h_4\left(h_2+N^2+1\right)}{-h_2^2+N^4-N^2+1},$$
$$g=-\frac{3h_4N^2}{-h_2^2+N^4-N^2+1},$$

under the constraint condition

$$h_4^2\left(-h_2-N^2-1\right)\left(9N^2-\left(-h_2-N^2-1\right)\left(h_2+2\left(-N^2-1\right)\right)\right)=0.$$

If $N\to 1$, then the soliton solution is obtained

$$u_{4,2}(x,t)=-\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\coth(tw-x)}{\sqrt{b}\sqrt{-\frac{h_4\left(\left(h_2+2\right)\operatorname{csch}^2(tw-x)+h_2-1\right)}{h_2^2-1}}}, \tag{27}$$

such that

$$h_4^2\left(-h_2-2\right)\left(9-\left(-h_2-2\right)\left(h_2-4\right)\right)=0.$$

If $N\to 0$, then the rational solution is obtained, such that

$$h_4^2\left(-h_2-1\right)\left(-\left(-h_2-1\right)\left(h_2-2\right)\right)=0.$$

5. Assuming $l_0=-N^2$, $l_2=2N^2-1$, $l_4=1-N^2$, where $0<N<1$, the function $Q(\mu)=nc(\mu,N)$ is determined, leading to:

$$u_{5,1}(x,t)=\alpha_1\left(\frac{nc(\mu,N)}{\sqrt{f\left(nc(\mu,N)\right)^2+g}}\right), \tag{28}$$

where the values of f and g are ascertained by

$$f=\frac{h_4\left(h_2-2N^2+1\right)}{-h_2^2+N^4-N^2+1},$$

$$g = \frac{3h_4N^2}{-h_2^2 + N^4 - N^2 + 1},$$

under the constraint condition

$$h_4^2\left(-h_2 + 2N^2 - 1\right)\left(-\left(-h_2 + 2N^2 - 1\right)\left(h_2 + 2\left(2N^2 - 1\right)\right) - 9\left(1 - N^2\right)N^2\right) = 0.$$

If $N \to 1$, then the dark soliton solution is obtained

$$u_{5,2}(x,t) = \frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\cosh(tw-x)}{\sqrt{b}\sqrt{\frac{h_4\left(-h_2\cosh^2(tw-x)+\cosh^2(tw-x)-3\right)}{h_2^2-1}}}, \tag{29}$$

such that

$$h_4^2\left(1 - h_2\right)\left(-\left(1 - h_2\right)\left(h_2 + 2\right)\right) = 0.$$

If $N \to 0$, then the rational solution is obtained, such that

$$h_4^2\left(-h_2 - 1\right)\left(-\left(-h_2 - 1\right)\left(h_2 - 2\right)\right) = 0.$$

6. Assuming $l_0 = -1$, $l_2 = 2 - N^2$, $l_4 = -(1 - N^2)$, where $0 < N < 1$, the function $Q(\mu) = nd(\mu, N)$ is obtained, resulting in:

$$u_{6,1}(x,t) = \alpha_1\left(\frac{nd(\mu,N)}{\sqrt{f\left(nd(\mu,N)\right)^2 + g}}\right), \tag{30}$$

where the values of f and g are ascertained by

$$f = \frac{h_4\left(h_2 + N^2 - 2\right)}{-h_2^2 + N^4 - N^2 + 1},$$
$$g = \frac{3h_4}{-h_2^2 + N^4 - N^2 + 1},$$

subject to the specified condition

$$h_4^2\left(-h_2 - N^2 + 2\right)\left(-\left(-h_2 - N^2 + 2\right)\left(h_2 + 2\left(2 - N^2\right)\right) - 9\left(N^2 - 1\right)\right) = 0.$$

As N approaches 1, the singular solution is obtained.

$$u_{6,2}(x,t) = \frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\cosh(tw-x)}{\sqrt{b}\sqrt{\frac{h_4\left(-h_2\cosh^2(tw-x)+\cosh^2(tw-x)-3\right)}{h_2^2-1}}}, \tag{31}$$

such that

$$h_4^2\left(1 - h_2\right)\left(-\left(1 - h_2\right)\left(h_2 + 2\right)\right).$$

7. If $l_0 = 1$, $l_2 = 2 - N^2$, $l_4 = 1 - N^2$, $0 < N < 1$, then $Q(\mu) = sc(\mu, N)$, resulting in

$$u_{7,1}(x,t) = \alpha_1\left(\frac{sc(\mu,N)}{\sqrt{f\left(sc(\mu,N)\right)^2 + g}}\right), \tag{32}$$

where the values of f and g are ascertained by

$$f = \frac{h_4\left(h_2 + N^2 - 2\right)}{-h_2^2 + N^4 - N^2 + 1},$$
$$g = -\frac{3h_4}{-h_2^2 + N^4 - N^2 + 1},$$

subject to the specified condition

$$h_4^2\left(-h_2 - N^2 + 2\right)\left(9\left(1 - N^2\right) - \left(-h_2 - N^2 + 2\right)\left(h_2 + 2\left(2 - N^2\right)\right)\right) = 0.$$

If $N \to 1$, then the soliton solution is obtained

$$u_{7,2}(x,t) = -\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\sinh(tw-x)}{\sqrt{b}\sqrt{\frac{h_4\left(-h_2\sinh^2(tw-x)+\sinh^2(tw-x)+3\right)}{h_2^2-1}}}, \tag{33}$$

such that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right).$$

As N tends toward 0, the solution becomes a periodic wave

$$u_{7,3}(x,t)=-\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\tan(tw-x)}{\sqrt{b}\sqrt{\frac{h_4\left(3-\left(h_2-2\right)\tan^2(tw-x)\right)}{h_2^2-1}}}, \tag{34}$$

such that

$$h_4^2\left(2-h_2\right)\left(9-\left(2-h_2\right)\left(h_2+4\right)\right).$$

8. If $l_0=1$, $l_2=2N^2-1$, $l_4=-N^2\left(1-N^2\right)$, $0<N<1$, then $Q(\mu)=sd(\mu,N)$ resulting in

$$u_{8,1}(x,t)=\alpha_1\left(\frac{sd(\mu,N)}{\sqrt{f\left(sd(\mu,N)\right)^2+g}}\right), \tag{35}$$

where the values of f and g are ascertained by

$$f=\frac{h_4\left(h_2-2N^2+1\right)}{-h_2^2+N^4-N^2+1},$$
$$g=-\frac{3h_4}{-h_2^2+N^4-N^2+1},$$

under the constraint condition

$$h_4^2\left(-h_2+2N^2-1\right)\left(-\left(-h_2+2N^2-1\right)\left(h_2+2\left(2N^2-1\right)\right)-9\left(1-N^2\right)N^2\right)=0.$$

If $N\to 1$, then the soliton solution is obtained

$$u_{8,2}(x,t)=-\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\sinh(tw-x)}{\sqrt{b}\sqrt{h_4\left(\frac{3}{h_2^2-1}-\frac{\sinh^2(tw-x)}{h_2+1}\right)}}, \tag{36}$$

such that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right).$$

9. If $l_0=1-N^2$, $l_2=2-N^2$, $l_4=1$, $0<N<1$, then $Q(\mu)=cs(\mu,N)$ and resulting in

$$u_{9,1}(x,t)=\alpha_1\left(\frac{cs(\mu,N)}{\sqrt{f\left(cs(\mu,N)\right)^2+g}}\right), \tag{37}$$

where the values of f and g are ascertained by

$$f=\frac{h_4\left(h_2+N^2-2\right)}{-h_2^2+N^4-N^2+1},$$
$$g=\frac{3h_4\left(N^2-1\right)}{-h_2^2+N^4-N^2+1},$$

under the constraint condition

$$h_4^2\left(-h_2-N^2+2\right)\left(9\left(1-N^2\right)-\left(-h_2-N^2+2\right)\left(h_2+2\left(2-N^2\right)\right)\right)=0.$$

If $N\to 1$, then the rational solution is obtained, such that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)=0.$$

As N approaches 0, the solution becomes a periodic wave.

$$u_{9,3}(x,t)=-\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\cot(tw-x)}{\sqrt{b}\sqrt{\frac{h_4\left(-\left(h_2-2\right)\csc^2(tw-x)+h_2+1\right)}{h_2^2-1}}}, \tag{38}$$

such that

$$h_4^2\left(2-h_2\right)\left(9-\left(2-h_2\right)\left(h_2+4\right)\right)=0.$$

The solutions are obtained under some restrictive conditions. for instance in Eq. (38) the condition $h_2^2 \neq 1$ or $b \neq 1$ must be satisfied, also functions such as tangent and cotangent have the property of discontinuity at some points, this property can be used to examine the shock wave structure, which is known as "blow-up" in soliton dynamics [56,57].

10. If $l_0 = -N^2\left(1-N^2\right)$, $l_2 = 2N^2-1$, $l_4 = 1$, $0 < N < 1$, then $Q(\mu) = ds(\mu, N)$ and resulting in

$$u_{10}(x,t) = \alpha_1\left(\frac{ds(\mu,N)}{\sqrt{f\left(ds(\mu,N)\right)^2+g}}\right), \tag{39}$$

where the values of f and g are ascertained by

$$f = \frac{h_4\left(h_2-2N^2+1\right)}{-h_2^2+N^4-N^2+1},$$
$$g = -\frac{3h_4N^2\left(N^2-1\right)}{-h_2^2+N^4-N^2+1},$$

under the constraint condition

$$h_4^2\left(-h_2+2N^2-1\right)\left(-\left(-h_2+2N^2-1\right)\left(h_2+2\left(2N^2-1\right)\right)-9\left(1-N^2\right)N^2\right)=0.$$

11. If $l_0 = \frac{1-N^2}{4}$, $l_2 = \frac{1+N^2}{2}$, $l_4 = \frac{1-N^2}{4}$, $0 < N < 1$, then $Q(\mu) = nc(\mu, N) \pm sc(\mu, N)$ or $Q(\mu) = \frac{cn(\mu,N)}{1\pm sn(\mu,N)}$ and resulting in

$$u_{11,0}(x,t) = \alpha_1\left(\frac{nc(\zeta,N)\pm sc(\zeta,N)}{\sqrt{f\left(nc(\zeta,N)\pm sc(\zeta,N)\right)^2+g}}\right), \tag{40}$$

or

$$u_{11,1}(x,t) = \alpha_1\left(\frac{\frac{cn(\mu,N)}{1\pm sn(\mu,N)}}{\sqrt{f\left(\frac{cn(\mu,N)}{1\pm sn(\mu,N)}\right)^2+g}}\right), \tag{41}$$

where the values of f and g are ascertained by

$$f = -\frac{8h_4\left(-2h_2+N^2+1\right)}{-16h_2^2+N^4+14N^2+1},$$
$$g = \frac{12h_4\left(N^2-1\right)}{-16h_2^2+N^4+14N^2+1},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(N^2+1\right)-h_2\right)\left(\frac{9}{16}\left(1-N^2\right)^2-\left(\frac{1}{2}\left(N^2+1\right)-h_2\right)\left(h_2+N^2+1\right)\right)=0.$$

If $N \to 1$, then the rational solution is obtained, such that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)=0.$$

As N tends toward 0, the solution becomes combined periodic waves

$$u_{11,2}(x,t) = \frac{\sqrt{h_4(c-w)(c+w)}(\sec(tw-x)-\tan(tw-x))}{\sqrt{2}\sqrt{b}\sqrt{\frac{h_4\left(4h_2(\sin(tw-x)-1)+\sin(tw-x)+5\right)}{\left(16h_2^2-1\right)(\sin(tw-x)+1)}}}, \tag{42}$$

or

$$u_{11,3}(x,t) = -\frac{\sqrt{h_4(c-w)(c+w)}\cos(tw-x)}{\sqrt{2}\sqrt{b}(\sin(tw-x)-1)\sqrt{\frac{h_4\left(\frac{2\left(1-2h_2\right)\cos^2(tw-x)}{(\sin(tw-x)-1)^2}+3\right)}{16h_2^2-1}}}, \tag{43}$$

are obtained, such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)\left(h_2+1\right)\right)=0.$$

12. If $l_0 = \frac{-\left(1-N^2\right)^2}{4}$, $l_2 = \frac{1+N^2}{2}$, $l_4 = \frac{-1}{4}$, $0 < N < 1$, then $Q(\mu) = Ncn(\mu, N) \pm dn(\mu, m)$ and resulting in

$$u_{12,1}(x,t) = \alpha_1\left(\frac{Ncn(\mu,N)\pm dn(\mu,N)}{\sqrt{f\left(Ncn(\mu,N)\pm dn(\mu,N)\right)^2+g}}\right), \tag{44}$$

where the values of f and g are ascertained by

$$f = -\frac{8h_4\left(-2h_2+N^2+1\right)}{-16h_2^2+N^4+14N^2+1},$$
$$g = \frac{12h_4\left(N^2-1\right)^2}{-16h_2^2+N^4+14N^2+1},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(N^2+1\right)-h_2\right)\left(\frac{9}{16}\left(1-N^2\right)^2-\left(\frac{1}{2}\left(N^2+1\right)-h_2\right)\left(h_2+N^2+1\right)\right)=0.$$

If $N \to 1$, then the rational solution is obtained, such that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)=0.$$

As N tends toward 0, the solution becomes periodic waves

$$u_{12,1}(x,t) = \frac{\sqrt{h_4(c-w)(c+w)}\cos(tw-x)}{\sqrt{2}\sqrt{b}\sqrt{\frac{h_4\left(-4h_2\cos^2(tw-x)+\cos(2tw-2x)-2\right)}{16h_2^2-1}}}, \tag{45}$$

where $\mu = x - wt$, such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)\left(h_2+1\right)\right)=0.$$

13. If $l_0 = \frac{1}{4}$, $l_2 = \frac{1-2N^2}{2}$, $l_4 = \frac{1}{4}$, $0 < N < 1$, then $Q(\mu) = \frac{sn(\mu,N)}{1\pm cn(\mu,N)}$ and resulting in

$$u_{13,1}(x,t) = \alpha_1\left(\frac{\frac{sn(\mu,N)}{1\pm cn(\mu,N)}}{\sqrt{f\left(\frac{sn(\mu,N)}{1\pm cn(\mu,N)}\right)^2+g}}\right), \tag{46}$$

where the values of f and g are ascertained by

$$f = \frac{8h_4\left(2h_2+2N^2-1\right)}{-16h_2^2+16N^4-16N^2+1},$$
$$g = -\frac{12h_4}{-16h_2^2+16N^4-16N^2+1},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1-2N^2\right)-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}\left(1-2N^2\right)-h_2\right)\left(h_2-2N^2+1\right)\right)=0.$$

If $N \to 1$, then the following solution is derived

$$u_{13,2}(x,t) = -\frac{\sqrt{h_4(c-w)(c+w)}\tanh\left(\frac{1}{2}(tw-x)\right)}{\sqrt{2}\sqrt{b}\sqrt{\frac{h_4\left(\frac{8h_2+4}{\cosh(tw-x)+1}-4h_2+1\right)}{16h_2^2-1}}}, \tag{47}$$

such that

$$h_4^2\left(-h_2-\frac{1}{2}\right)\left(\frac{9}{16}-\left(-h_2-\frac{1}{2}\right)\left(h_2-1\right)\right)=0.$$

As N tends toward 0, the solution becomes combined periodic waves

$$u_{13,3}(x,t) = -\frac{\sqrt{h_4(c-w)(c+w)}\tan\left(\frac{1}{2}(tw-x)\right)}{\sqrt{2}\sqrt{b}\sqrt{\frac{h_4\left(\frac{4-8h_2}{\cos(tw-x)+1}+4h_2+1\right)}{16h_2^2-1}}}, \tag{48}$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)\left(h_2+1\right)\right)=0.$$

14. If $l_0 = \frac{1}{4}$, $l_2 = \frac{1+N^2}{2}$, $l_4 = \frac{\left(1-N^2\right)^2}{4}$, $0 < N < 1$, then $Q(\mu) = \frac{sn(\mu,N)}{cn(\mu,N)\pm dn(\mu,N)}$ and resulting in

$$u_{14,1}(x,t) = \alpha_1\left(\frac{\frac{sn(\mu,N)}{cn(\mu,N)\pm dn(\mu,N)}}{\sqrt{f\left(\frac{sn(\mu,N)}{cn(\mu,N)\pm dn(\mu,N)}\right)^2+g}}\right), \tag{49}$$

where the values of f and g are ascertained by

$$f = -\frac{8h_4\left(-2h_2+N^2+1\right)}{-16h_2^2+N^4+14N^2+1},$$
$$g = -\frac{12h_4}{-16h_2^2+N^4+14N^2+1},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(N^2+1\right)-h_2\right)\left(\frac{9}{16}\left(1-N^2\right)^2-\left(\frac{1}{2}\left(N^2+1\right)-h_2\right)\left(h_2+N^2+1\right)\right)=0.$$

If $N \to 1$, then the soliton solution is obtained

$$u_{14,2}(x,t) = -\frac{\sqrt{2}\sqrt{h_4(c-w)(c+w)}\sinh(tw-x)}{\sqrt{b}\sqrt{\frac{h_4\left(-h_2\sinh^2(tw-x)+\sinh^2(tw-x)+3\right)}{h_2^2-1}}}, \tag{50}$$

such that

$$h_4^2\left(1-h_2\right)\left(-\left(1-h_2\right)\left(h_2+2\right)\right)=0.$$

As N tends toward 0, the solution becomes periodic wave

$$u_{14,3}(x,t) = -\frac{\sqrt{h_4(c-w)(c+w)}\tan\left(\frac{1}{2}(tw-x)\right)}{\sqrt{2}\sqrt{b}\sqrt{\frac{h_4\left(\frac{4-8h_2}{\cos(tw-x)+1}+4h_2+1\right)}{16h_2^2-1}}}, \tag{51}$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left(\frac{9}{16}-\left(\frac{1}{2}-h_2\right)\left(h_2+1\right)\right)=0.$$

4. Results and discussions

In this study, we aim to generate traveling wave solutions for the (1 + 1) dimensional Klein–Gordon equation using the φ^6-model expansion method and evaluate the effectiveness of this method for understanding the dynamics of solitons. Our findings unequivocally demonstrate the strengths and application potential of the φ^6-model expansion method in solving the complex nonlinear dynamics of the Klein–Gordon equation. The φ^6- model expansion method offers both a simple and accurate approach to solving nonlinear equations and is highly successful in obtaining soliton solutions. The fact that this method accurately produces solutions of the Klein–Gordon equation indicates that it can also provide reliable solutions for more complex equations. Also, the method offers a significant advantage over other methods by providing the advantage of high accuracy and computational speed. One of the most remarkable findings of the study is the obtaining of solutions of kink-type solitons, known as topological solitons, especially in the context of water waves. The fact that the velocities of the generated solitons are independent of the wave amplitude indicates that the findings contribute to a better understanding of nonlinear dynamics. In addition, the solutions obtained include dark, singular, periodic and composite singular soliton solutions. This diversity provides the opportunity to study the dynamical properties of the Klein–Gordon equation in a wide spectrum and to investigate its applications in physical processes in depth.

Whether the solutions tested in this study accurately represent physical mechanisms will be one of the key discussion points of the research. In particular, the extent to which the results obtained through nonlinear differential equations are compatible with the real-world behavior of the modeled systems should be carefully examined. The theoretical accuracy of such mathematical models will be assessed as well as whether they accurately capture the dynamics of physical processes. In this context, it is not enough for the solutions to be mathematically consistent; they must also be consistent with physical phenomena. This aspect of the study is critical in determining the validity and applicability of the model and will have a direct impact on the reliability of the results. Studying the behavior of the systems with appropriate parameter values in the generated traveling wave solutions will provide important insights into the dynamics of the systems. Analyzing the effects of parameter changes on the system helps us to understand under which conditions certain behaviors occur and allows us to establish the initial conditions of the system. In addition to revealing the fundamental properties of complex systems, such analyses increase the controllability and predictability of systems. Especially in nonlinear systems, such an investigation plays an important role in understanding the overall dynamical structure of the system, given that even small parameter changes can have significant and sometimes unexpected consequences. A detailed analysis across the parameter space will shed light on the physical mechanisms.

In this study, Eq. (20), which reflects the periodic soliton property, will be analyzed from the traveling wave solutions obtained for the discussion. Periodic wave structures that repeat in a specific pattern can be a powerful tool for understanding and modeling the complex structure of Rayleigh waves. It was the British physicist Lord Rayleigh who introduced Rayleigh waves to the literature and mathematically modeled them in the 4th quarter of the 19th century [58]. Rayleigh waves are also known as surface waves because they propagate on the ground surface and exhibit semi-circular motions. They are generated as a result of seismic events

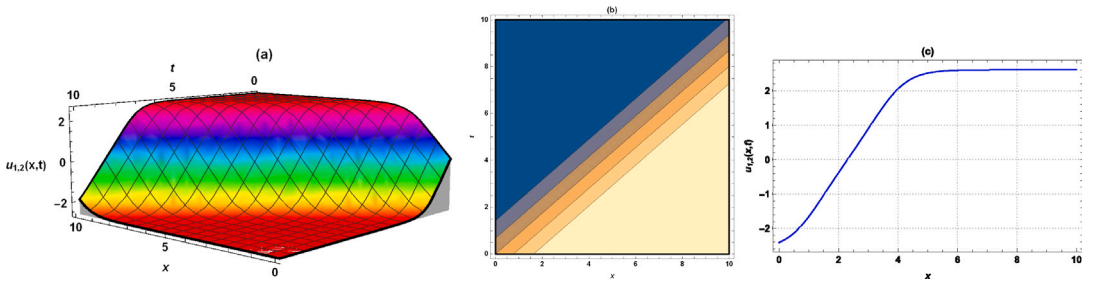


Fig. 1. The visual depiction of the solution provided in Eq. (18), particularly as N approaches 1, (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.5,\ c = 0.1,\ h_2 = 0.3,\ h_4 = 0.1,\ w = 1.15$.

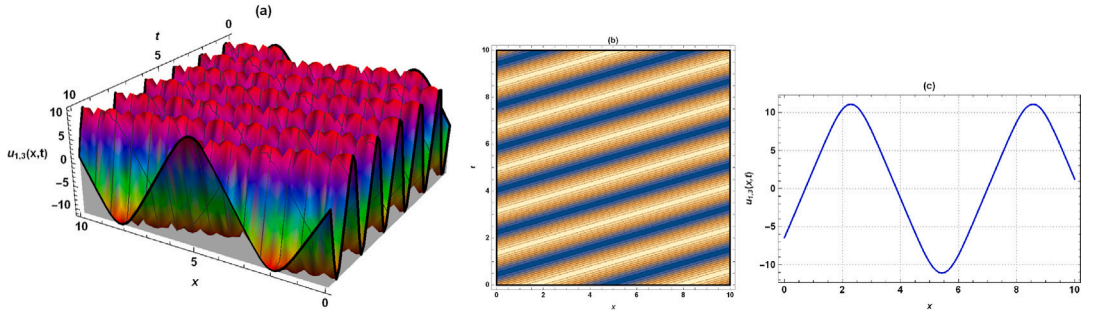


Fig. 2. The visual depiction of the solution provided in Eq. (19), particularly as N approaches 0, ; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.1,\ c = 0.01,\ h_2 = 0.01,\ h_4 = 0.1,\ w = 3.5$.

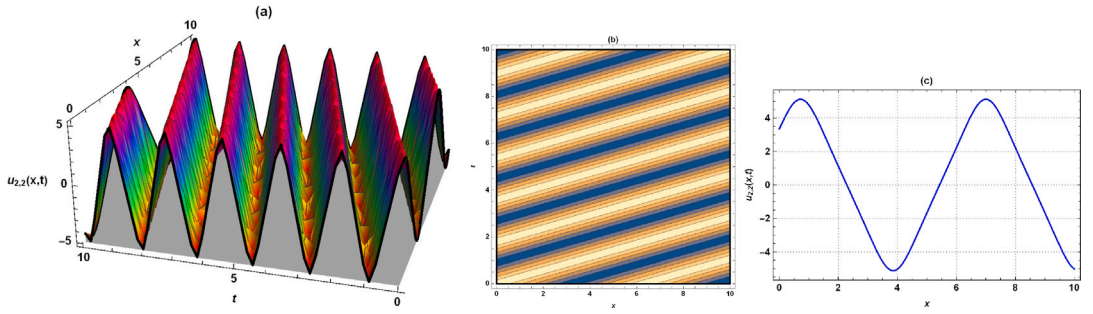


Fig. 3. The visual depiction of the solution provided in Eq. (21), particularly as N approaches 1; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.5,\ c = 0.1,\ h_2 = 0.3,\ h_4 = 0.1,\ w = 3.5$.

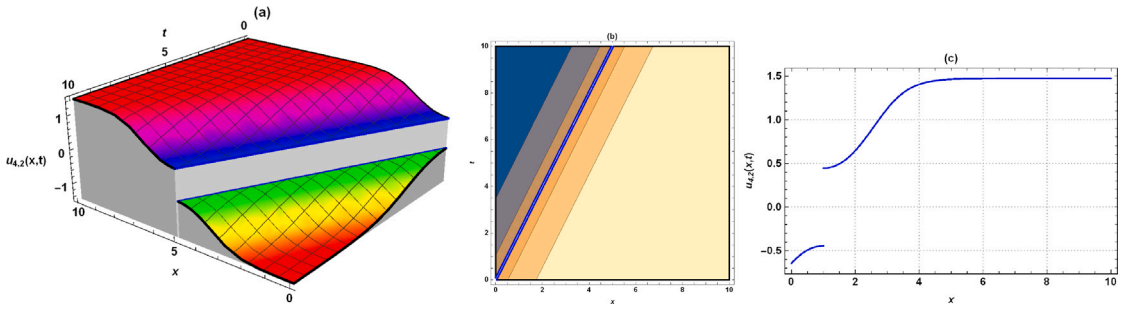


Fig. 4. The visual depiction of the solution provided in Eq. (27), particularly as N approaches 1; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.5,\ c = 0.12,\ h_2 = 1.3,\ h_4 = 0.2,\ w = 0.5$.

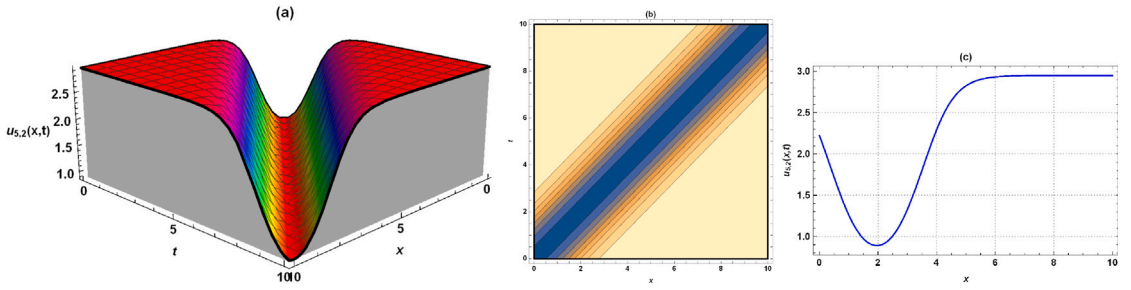


Fig. 5. The visual depiction of the solution provided in Eq. (29), particularly as N approaches 1; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.5,\ c = 0.12,\ h_2 = 1.3,\ h_4 = 0.2,\ w = 0.98$.

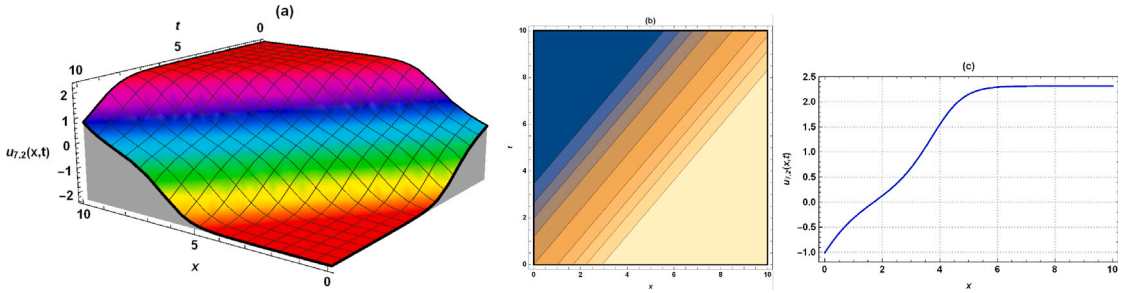


Fig. 6. The visual depiction of the solution provided in Eq. (33), particularly as N approaches 1; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.5,\ c = 0.12,\ h_2 = 0.9,\ h_4 = 0.2,\ w = 0.85$.

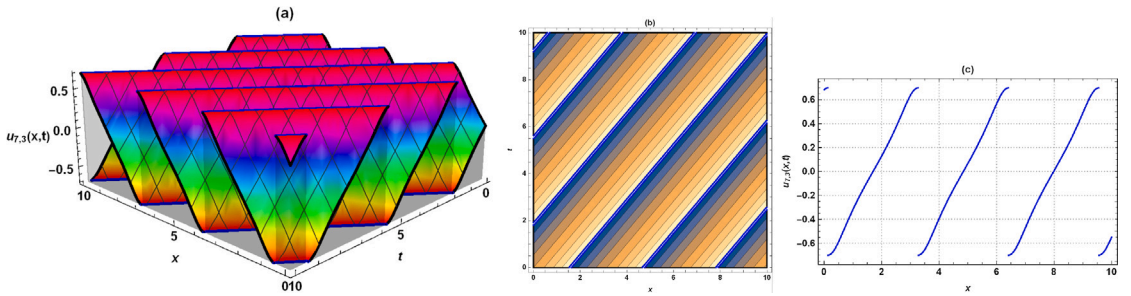


Fig. 7. The visual depiction of the solution provided in Eq. (34), particularly as N approaches 1; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.5,\ c = 0.12,\ h_2 = 0.9,\ h_4 = 0.2,\ w = 0.85$.

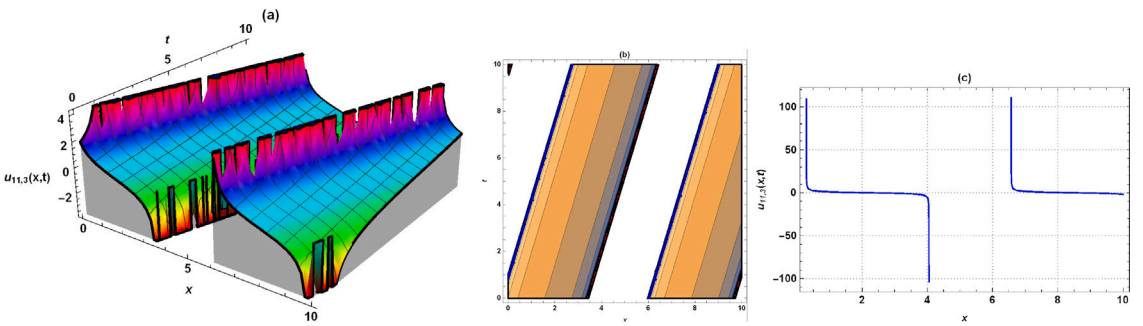


Fig. 8. The visual depiction of the solution provided in Eq. (43), particularly as N approaches 0; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.3,\ c = 0.6,\ h_2 = 0.9,\ h_4 = 0.2,\ w = 0.3$.

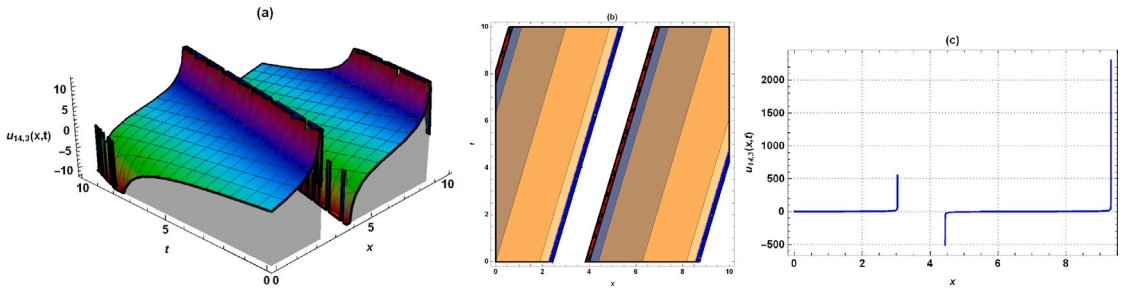


Fig. 9. The visual depiction of the solution provided in Eq. (51), particularly as N approaches 0; (a) is the 3D graphic, (b) is the contour graphic while (c) is the 2D graphic for $b = 0.2,\ c = 1.6,\ h_2 = 0.6,\ h_4 = 0.1,\ w = 0.3$.

and propagate across the earth's surface, carrying the effects of large-scale earthquakes. Rayleigh waves are waves that roll along the surface of the earth like waves propagating in a puddle of water such as a lake or sea [59]. Periodic solitons, which represent a regular wave motion, can be associated with Rayleigh waves, which propagate along the surface and have a certain periodicity and repetitive structure. Therefore, periodic solitons can provide a suitable framework for the mathematical modeling of Rayleigh waves. Another name for periodic solitons is wave solutions, which have a certain momentum and energy. A direct connection between these two wave types may be established since Rayleigh waves likewise build a structure that transports energy over the surface.

In Eq. (20), t represents time and x represents space components. They are also the basic components of the wave propagation with respect to time and space. It is the physical parameter w that represents the wave frequency or angular frequency. This parameter is a frequency component that determines the periodic nature of the wave. The parameter c, which defines the propagation velocity and the structure in space that affects the propagation pattern and behavior of the wave, is included in Eq. (10), which models the physical mechanism. This constant is a physical parameter that depends on the elasticity and density of the medium. The term that affects the amplitude of the wave is $\sqrt{(c-w)(c+w)}$, which implies a non-linear relationship with the wave frequency and the properties of the medium in which the wave propagates. In the possible case of $c^2 - w^2 \leq 0$, it could cause the wave to lose its observability or be interpreted as a different type of wave. Therefore, the constraint condition can be set to $c - w > 0$. The term b, which describes the nonlinear properties and behaviors of Eq. (10) modeling the system, represents the nonlinear relationships and interactions in the system. The $Cos(tw - x)$ function takes over the periodic nature of the wave function and the role of the skeleton of the wave pattern. Since Rayleigh waves exhibit a periodic structure in a range or cross section along the surface, an important connection can be made with the periodic soliton presented by Eq. (20) in terms of the repetitive nature of the waveform, energy dissipation and analysis of amplitude variations. Moreover, Rayleigh waves are damped along the propagating medium and move perpendicular to the propagation direction. Therefore, periodic components, such as the cosine function, are suitable for modeling harmonic motions at the surface. Using the wave function presented by Eq. (20), numerical values of the parameters in the traveling wave solution can be assigned to model Rayleigh waves over a range and to visualize their periodic nature and their variation with position time. These assigned values will determine the initial conditions in the stability region of the system. The effects of different degrees of nonlinearity on the wave behavior can be presented as follows, provided that all parameters except $b > 0$, which represents the degree of nonlinearity, remain constant.

Since Rayleigh waves exhibit a periodic structure in a range or cross section along the surface, an important connection can be made with the periodic soliton presented by Eq. (20) in terms of the repetitive nature of the waveform, energy dissipation and analysis of amplitude variations. While constructing Fig. 10, the numerical values of the parameters in Eq. (20) have been determined by taking into account the propagation speed and frequency characteristics of the Rayleigh wave [60]. The most significant conclusion drawn from this debate is that slow surface waves, like Rayleigh waves, may have a physical significance associated with the numerical result $\sqrt{(c-w)(c+w)} = \sqrt{0.75}$ in the wave function. As a result, the characteristics of slow surface waves are compatible with the phase velocity being greater than the angular frequency. It also makes it possible to model a Rayleigh wave, which is influenced by the material's elastic properties and transports energy on the surface. As seen in Fig. 10, as the numerical value of the b parameter representing the degree of nonlinearity in Eq. (20) modeling the wave increases, the decrease in wave amplitude can reflect various effects and results in a physical system. This indicates that the dynamics of the system are more complex, while at the same time nonlinear effects can affect the wave motion in a weakening way. An increase in the numerical value of parameter b may lead to a stronger likelihood of nonlinearity, a drop in the wave amplitude, which is the response of the wave solution, and a greater dissipation of the wave's energy. It may also be explained by the wave's energy being somewhat attenuated. This plays an important role in fundamental processes in Rayleigh wave propagation, such as energy transfer and wave shape preservation.

To understand that Eq. (20), presented for the first time in this work and nominated as a model of Rayleigh waves, reflects periodic properties, the following equation can be analyzed to study how it behaves with respect to time and space. For this analysis, the following expression can be obtained by considering the parameter and $b = 0.1$ values used in the construction of Fig. 10, which describes the wave pattern and energy level.

$$-1.87083 < \lim_{\substack{x\to\infty \\ t\to\infty}} u_{2,2}(x,t) < 1.87083. \tag{52}$$

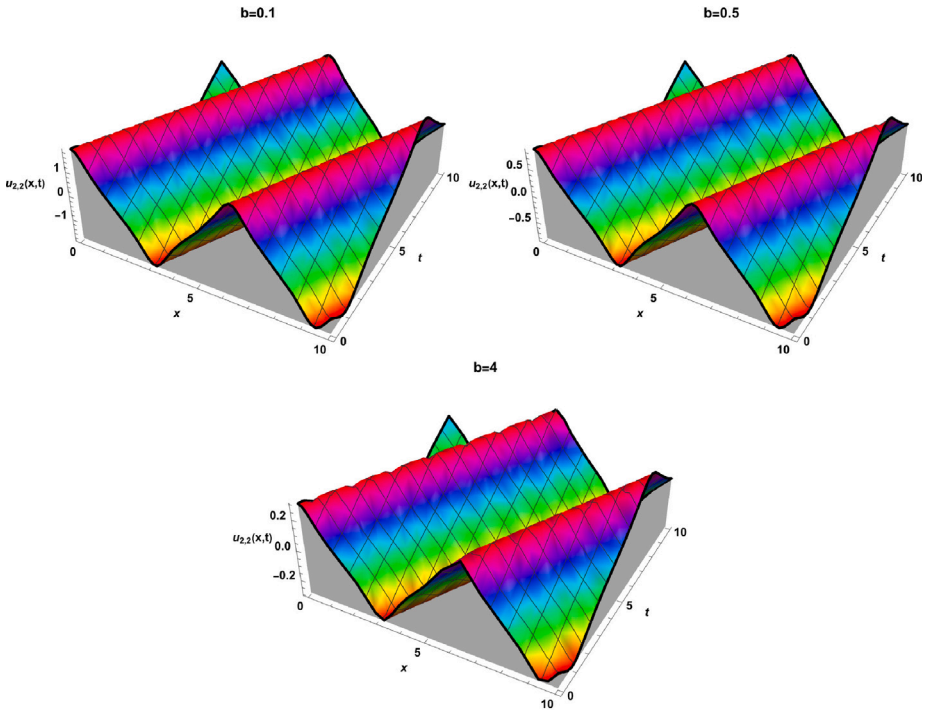


Fig. 10. Surface wave behavior, taking into account the nonlinearities and behavior of the traveling wave solution presented by Eq. (20) for $c = 1.6$, $h_2 = 1.1$, $w = 0.5$.

One of the most obvious characteristics of Rayleigh waves being surface waves is that most of the energy is concentrated at or near the surface. This means that the energy of the wave is concentrated at the surface rather than at depth and affects the propagation, pattern and energy distribution of the wave along the surface. The periodic characteristic of Rayleigh waves allows them to form a pattern that repeats at regular intervals during energy transmission. This periodic pattern propagating along the surface means that energy is continuously concentrated and dissipated near the surface. Under certain conditions and in the stability region, the numerical values given to the parameters indicate that the energy of the wave, represented by Eq. (52), is optimized within a certain range. As a result, it provides theoretical support for the periodic, singular and lone wave behavior of surface waves and can play an important role in explaining the dynamical properties of such waves. Furthermore, the introduced models are capable of modeling regular oscillations that occur on elastic surfaces, such as Rayleigh waves, and allow the wave to propagate stably at a given speed and optimize the energy transfer. It serves as an important theoretical instrument for modeling the complex behavior of surface waves under nonlinear effects. Traveling waves are types of waves that can propagate while maintaining their amplitude and speed. This property provides theoretical support that Rayleigh waves can exhibit lone wave behavior under certain boundary conditions.

5. Conclusion

We present the φ^6-model expansion technique, which is a novel and versatile method for solving nonlinear partial differential equations, including the Klein–Gordon equation. Numerous solutions are produced by this approach, including kink-type solitons with unique velocity properties, dark, solitary, periodic, and coupled singular solitons. By studying surface waves, particularly Rayleigh waves, and their motion and structure, we show how periodic behavior is essential to the propagation of waves. The method provides new insights into the nonlinear dynamics of physical systems, backed by realistic and visually appealing depictions. Our findings, visually depicted in Figs. 1–9, provide a comprehensive understanding of these solutions at different time intervals, elucidating their pivotal role in energy transmission across spatial domains. The dynamic interplay of these traveling waves, dictated by varying parameter values, introduces captivating alterations in their behavior over time.

A noteworthy aspect of our study is the prediction that the traveling wave behavior undergoes distinct changes for different values of each parameter. This insight not only enhances our comprehension of the Klein–Gordon equation's intricate dynamics but also sets the stage for further investigations into the physical properties governing nonlinear distribution dynamics. Beyond its immediate implications for this specific equation, the proposed φ^6-model expansion method emerges as a valuable and practical tool, offering a systematic approach to obtaining precise solutions for a diverse range of nonlinear partial differential equations. Thorough examination of surface wave characteristics, specifically concentrating on Rayleigh waves, highlights the significance of periodic behavior in shaping wave propagation characteristics.

As we conclude this study, we envision our results as a catalyst for future research endeavors, inspiring deeper inquiries into the complex interactions of parameters and their influence on the evolving behavior of nonlinear wave phenomena. Through this work, we contribute to the broader landscape of mathematical methods, advancing our understanding of intricate nonlinear dynamics and paving the way for innovative applications across various scientific disciplines.

CRediT authorship contribution statement

Asif Yokus: Writing – review & editing, Visualization, Supervision, Software, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **Muhammad Abubakar Isah:** Writing – review & editing, Visualization, Software, Methodology, Investigation, Formal analysis, Conceptualization.

Ethical standards

The writers guarantee that this work complies with ethical guidelines.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

This publication does not qualify for data sharing since no datasets were gathered or examined for this inquiry. The manuscript contains no data connected.

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